**RANDOM PROCESSES: 2004 EXAM SOLUTION.**

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**Problem 1.**

**(a)** The process $X$ is not Gaussian. Suppose it is, then e.g. if $d = 2$ and $r_1 = -r_2 = 1$, $p_1 = p_2 = 1/2$ then $EX_0X_1 = EX_0^2 Ea + EX_0 E\varepsilon_1 = 0$ implies that $X_0$ and $X_1$ are independent and thus

$$E\big(\varphi(X_1)|X_0\big) = E\varphi(X_1)$$

should hold for any function $\varphi$, such that $E|\varphi(X_1)| < \infty$. The latter fails for $\varphi(x) = x^2$:

$$E(X_1^2|X_0) = E(X_0^2a^2 + 2X_0a\varepsilon_1 + \varepsilon_1^2|X_0) = X_0^2 + 1.$$

**(b)** Given $a$, $X_n$ is a linear combination of $\varepsilon_i$'s, (which are independent of $a$!) and hence is conditionally Gaussian. Namely

$$X_n = a^nX_0 + \sum_{i=1}^n \varepsilon_i a^{n-i}, \tag{1.1}$$

i.e. $X_n = \alpha^n\varepsilon^n$, where $\alpha^n$ is the appropriate row vector, depending on $a$, and $\varepsilon^n$ is the vector with $\varepsilon_i$ as entries. Hence

$$E\big(\exp\{i\lambda\alpha^n\varepsilon^n\}|a\big) = \int_{\mathbb{R}^n}\exp\{i\lambda\alpha^n x\}\frac{1}{(2\pi)^{n/2}}\exp\left\{-\sum_i^n x_i^2/2\right\}dx_1...dx_n =$$

$$\exp\left\{-\frac{1}{2}\lambda^2\|\alpha^n\|^2\right\} \quad \text{(why is the last equality correct?)}$$

**(c)** Given $a$, $X$ is a conditionally Gaussian process. The proof is by similar arguments as in **(b)** ($\alpha^n$ is a matrix this time).

**(d)** $a$ is not Gaussian, conditioned on $X_0^n$. One may be tempted to give (wrong) positive answer, since given $X_0^n$, $a$ is a linear combinations of $\varepsilon_i$'s. Note however that this time $\varepsilon_i$'s depend on the condition $X_0^n$ ! Again if $d = 2$, $r_1 = -r_2 = 1$, $p_1 = p_2 = 1/2$, then e.g. $P(|a| \le 0.17|X_0^n) = 0$, so that $P(a \le x|X_0^n)$ cannot have Gaussian density. In fact the conditional distribution is lattice (discrete) as claimed below.

**(e)** By standard arguments $\pi_n(i) := G^i(X_0^{n-1}, X_n)$ should satisfy

$$E\Big(I(a = r_i)h(X_n)|X_0^{n-1}\Big) = E\Big(G^i(X_0^{n-1}, X_n)h(X_n)|X_0^{n-1}\Big)$$

*Date*: July, 9, 2004.

for any test function $h$. Then

$$E\Big(I(a=r_i)h(X_n)|X_0^{n-1}\Big) = E\Big(I(a=r_i)h(r_iX_{n-1}+\varepsilon_n)|X_0^{n-1}\Big) = \pi_{n-1}(i)\int_{\mathbb{R}}\psi(u-r_iX_{n-1})h(u)du$$

where $\psi(x)$ is the standard Gaussian density. Similar arguments for the right hand side of the above condition and arbitrariness of $h$ imply

$$\pi_n(i) = \frac{\pi_{n-1}(i)\psi(X_n - r_iX_{n-1})}{\sum_{j=1}^d \pi_{n-1}(j)\psi(X_n - r_jX_{n-1})}, \quad \pi_0(i) = p_i.$$

**(f)** No - by the same argument as in **(a)**.

**(g)** Yes - by the same argument as in **(b)**.

**(h)** Yes - by the same argument as in **(c)**.

**(i)** $a$ is conditionally Gaussian given $X_0^n$. The proof can be found in Problem 5.3 of home assignments.

**(j)** $X_n$ does not converge to zero in probability: fix any constant $-1 < r < 1$

$$P\big(|X_n|\ge\varepsilon\big) = EI\big(|X_n|\ge\varepsilon\big) \ge EI\big(|X_n|\ge\varepsilon\big)I\big(|a|\le r\big) = EP\big(|X_n|\ge\varepsilon|a\big)I\big(|a|\le r\big)$$

Recall that $X_n$ is conditionally Gaussian given $a$ with zero mean and conditional variance $V_n(a) = E(X_n^2|a)$, satisfying the recursion $V_n(a) = a^2V_{n-1}(a)+1$, subject to $V_0 = 1$. Note that $V_n(a)$ increases and $1 \le V_n(a) \le (1-r^2)^{-1}$ $P$-a.s. on the set $\{|a|\le r\}$ for all $n\ge 0$. In particular

$$P\big(|X_n|\ge\varepsilon|a\big) = \int_{|x|\ge\varepsilon}\frac{1}{\sqrt{2\pi V_n}}\exp\{-x^2/(2V_n)\}dx \ge 1 - \frac{2\varepsilon}{\sqrt{2\pi V_n(a)}} \ge 1 - \frac{2\varepsilon\sqrt{1-r^2}}{\sqrt{2\pi}} := c > 0$$

So we have

$$P\big(|X_n|\ge\varepsilon\big) \ge cP\big(|a|\le r\big) > 0, \quad \forall n$$

and clearly $P\big(|X_n|\ge\varepsilon\big) \not\to 0$ for any $\varepsilon>0$. Hence it does not converge neither in $\mathbb{L}^1$ nor $P$-a.s. The latter means that $P(\lim_{n\to\infty} X_n = 0) < 1$. In fact $P(\lim_{n\to\infty} X_n = 0) = 0$ in this case, since

$$\{\lim_{n\to\infty} X_n = 0\} \subseteq \{\lim_{n\to\infty}|X_n - aX_{n-1}| = 0\} = \{\lim_{n\to\infty}\varepsilon_n = 0\},$$

whereas the latter set obviously has zero probability (e.g. by Borel-Cantelli lemma).

It doesn't converge in law to zero, since then it would converge to zero in probability. In fact it doesn't converge in law at all (i.e. to any random variable) - why?

**(k)** The solution (1.1) implies $EX_n = 0$ and so $a$ is orthogonal to $X_n$ for any $n\ge 0$:

$$EaX_n = Ea^{n+1}EX_0 + \sum_{i=1}^n Ea^{n-i+1}E\varepsilon_i = 0.$$

Then obviously $\widehat{E}(a|X_0^n) = 0$ and $E(a - \widehat{a}_n)^2 = Ea^2 = 1$.

**(l)** If $a$ is Gaussian, then $X$ is *conditionally* Gaussian, meaning that the generalized Kalman filter generates the required conditional expectation (see Problem 5.3 in home assignments). The filtering model is

$$\begin{aligned} a_n &= a_{n-1} \\ X_n &= X_{n-1}a_{n-1} + \varepsilon_n \end{aligned}$$

subject to $(a, X_0)$. The filtering equations for $\bar{a} = E(a|X_0^n) = E(a_n|X_0^n)$ and $\bar{P}_n = E\big[(a - \bar{a}_n)^2\big|X_0^n\big]$ are

$$\begin{aligned} \bar{a}_n &= \bar{a}_{n-1} + \frac{X_{n-1}\bar{P}_{n-1}}{X_{n-1}^2\bar{P}_{n-1} + 1}\big(X_n - X_{n-1}\bar{a}_{n-1}\big) \\ \bar{P}_n &= \bar{P}_{n-1} - \frac{X_{n-1}^2\bar{P}_{n-1}^2}{X_{n-1}^2\bar{P}_{n-1} + 1} \end{aligned}$$

subject to $\bar{a}_0 = 0$ and $\bar{P} = 1$. Note that the obtained estimate is nonlinear, as expected.

**(m)** Let $Q_n = \bar{P}_n^+$ (i.e. $Q_n = I(\bar{P}_n > 0)/\bar{P}_n$). Then

$$Q_n = Q_{n-1} + X_{n-1}^2 = 1 + \sum_{i=1}^n X_{i-1}^2.$$

or

$$\bar{P}_n = \frac{1}{1 + \sum_{i=1}^n X_{i-1}^2}, \quad n \geq 1.$$

Intuitively $\bar{P}_n$ should converge to zero, since the denominator grows to infinity. We claim that $\bar{P}_n$ converges to zero $P$-a.s. and thus also in probability and in law. Since $\bar{P}_n \leq 1$, this implies convergence in $\mathbb{L}^p$, $p \geq 1$.

Note that

$$\{\lim_{n\to\infty} \bar{P}_n \neq 0\} = \{\lim_{n\to\infty} \sum_{i=1}^n X_i^2 < \infty\} \subseteq \{\lim_{n\to\infty} X_n = 0\}.$$

Since by **(j)** $P(\lim_{n\to\infty} X_n = 0) = 0$, we conclude that $P(\lim_{n\to\infty} \bar{P}_n \neq 0) = 0$.

*Remark:* convergence $P$-a.s. roughly means that the square error converges to zero on any (possible) trajectory of $X_n$, while convergence in $\mathbb{L}^1$ means that the mean square error goes to zero.

**Problem 2.**

**(a)** Taking expectation from both sides of

$$\begin{aligned} dx_t &= y_t dt \\ dy_t &= -x_t dt - 2y_t(\beta dt + \sigma dW_t), \end{aligned}$$

one gets

$$\begin{aligned} d\bar{x}_t &= \bar{y}_t dt \\ d\bar{y}_t &= -\bar{x}_t dt - 2\beta\bar{y}_t dt, \end{aligned}$$

which means that on average the system behaves as the unperturbed pendulum for any $\beta > 0$. Since the initial conditions have zero mean, $Ex_t = 0$ and $Ey_t = 0$ for any $t \geq 0$.

**(b)** By the Ito formula

$$
\begin{aligned}
dq_t =& d(x_t^2) = 2x_t dx_t = 2x_t y_t dt = 2u_t dt \\
dr_t =& d(y_t^2) = 2y_t\Big( - x_t dt - 2y_t(\beta dt + \sigma dW_t)\Big) + 4\sigma^2 y_t^2 dt = \\
& \quad 4(\sigma^2-\beta)y_t^2 dt - 2x_t y_t dt - 4y_t^2\sigma dW_t = 4(\sigma^2-\beta)r_t dt - 2u_t dt - 4r_t\sigma dW_t \\
du_t =& d(x_t y_t) = y_t dx_t + x_t dy_t = y_t^2 dt - x_t^2 dt - 2y_t x_t(\beta dt + \sigma dW_t) = \\
& \quad r_t dt - q_t dt - 2\beta u_t dt - 2u_t \sigma dW_t
\end{aligned}
$$

**(c)** Taking the expectation we get

$$
\begin{aligned}
d\bar{q}_t =& 2\bar{u}_t dt \\
d\bar{r}_t =& 4(\sigma^2-\beta)\bar{r}_t dt - 2\bar{u}_t dt \\
d\bar{u}_t =& \bar{r}_t dt - \bar{q}_t dt - 2\beta\bar{u}_t dt
\end{aligned}
$$

or in matrix notation for $Z_t = (\bar{q}_t, \bar{r}_t, \bar{u}_t)$

$$
\dot{Z}_t = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 4(\sigma^2-\beta) & -2 \\ -1 & 1 & -2\beta \end{pmatrix} Z_t
$$

subject to $Z_0 = (1, 1, 0)$.

**(d)** The latter is a linear system with constant coefficients and its stability is completely determined by the eigenvalues of the coeff. matrix. Let $z = 4(\sigma^2 - \beta)$ for brevity. First note that $z = 0$ ($\sigma = \sqrt{\beta}$)is a "suspicious" point, for which the system has zero eigenvalue and thus is unstable.

The eigenvalues are the roots of the polynomial

$$
p(\lambda) = \det\begin{pmatrix} -\lambda & 0 & 2 \\ 0 & z-\lambda & -2 \\ -1 & 1 & -2\beta-\lambda \end{pmatrix} =
$$

$$
-\lambda\Big\{-(z-\lambda)(2\beta+\lambda)+2\Big\} + 2(z-\lambda) = \lambda\Big\{(z-\lambda)(2\beta+\lambda)-2\Big\} + 2(z-\lambda)
$$

Assume that $z > 0$, then

$$
\begin{aligned}
p(\lambda) = \lambda\Big\{(|z|-\lambda)(2\beta+\lambda)-2\Big\} + 2(|z|-\lambda) = \\
\lambda\Big\{(|z|-\lambda)(2\beta+\lambda)\Big\} + 2(|z|-2\lambda) := p_1(\lambda) - p_2(\lambda)
\end{aligned}
$$

The polynomial $p_1(\lambda)$ has roots at $0, |z|, -2\beta < 0$ and decreases (increases) to $-\infty$ ($\infty$) as $\lambda \to \infty$ ($\lambda \to -\infty$). The line $p_2(\lambda) = -2(|z| - 2\lambda)$ crosses the abscissa axis at $|z|/2 > 0$ and has a positive slope. Hence it crosses $p_1(\lambda)$ at some $\lambda^* \in (|z|/2, |z|)$ - see Figure. So the system has a positive root and is unstable for all $z > 0$, i.e. for all $\sigma^2 \ge \beta$. By the same arguments it is clear that the real root $\lambda^*$ is negative when $z < 0$ and $\lambda^* \in (z/2, 0)$. Expand the polynomial $p(\lambda)$, when $z < 0$, i.e. $z = -|z|$

$$
-p(\lambda) = \lambda^3 - \lambda^2(z-2\beta) - \lambda(2\beta z - 4) - 2z = \lambda^3 + \lambda^2(|z|+2\beta) + \lambda(2\beta|z|+4) + 2|z|
$$

Assume that the other two roots are complex (conjugate), denoted by $\lambda'$ and $\bar{\lambda}'$ and recall that

$$
-b_1 = \lambda^* + 2\Re\{\lambda'\}
$$

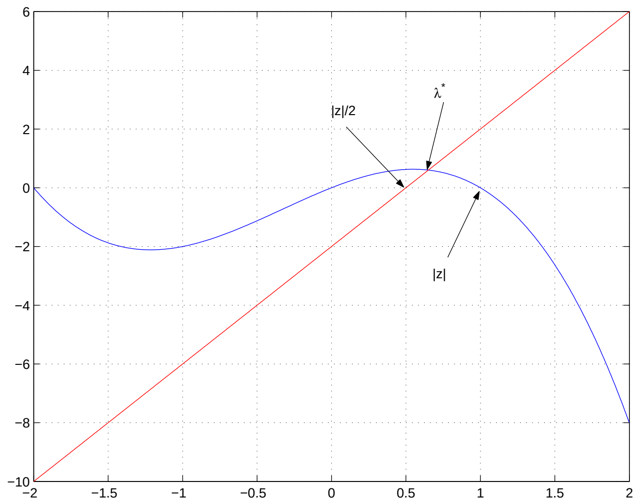


FIGURE 1. Case $z > 0$

where $b_1$ is the polynomial coefficient corresponding to power two. Since $\lambda^* < 0$

$$-(|z| + 2\beta) = -|\lambda^*| + 2\Re\{\lambda'\} \implies 2\Re\{\lambda'\} = -(|z| + 2\beta) + |\lambda^*|$$

Recall that $\lambda^* \in (-|z|/2, 0)$ in this case, so $\Re\{\lambda'\} \leq -\beta - |z|/2 < 0$, i.e. the system is stable.

Suppose now that the roots $\lambda_1$ and $\lambda_2$ are real. Recall that

$$-b_3 = \lambda^* \lambda_1 \lambda_2 = -|\lambda^*| \lambda_1 \lambda_2$$

where $b_3 = 2|z|$ is the free coefficient. Then $\lambda_1$ and $\lambda_2$ have the same sign. Using again

$$-b_1 = \lambda^* + \lambda_1 + \lambda_2 \implies \lambda_1 + \lambda_2 = -(|z| + 2\beta) + |\lambda^*|$$

we conclude that $\lambda_1 + \lambda_2 < -2\beta - |z|$, and thus both roots are negative.

In summary, $\sigma \geq \sqrt{\beta}$ destabilizes the pendulum, while for $\sigma < \sqrt{\beta}$ it remains stable.

*Remark:* the same conclusion may be obtained faster via Routh-Hurwitz Theorem.

*Remark:* a much more delicate analysis shows that $\sigma$ may exceed $\sqrt{\beta}$, while stability is preserved in e.g. $P$-a.s. sense.