STOCHASTIC PROCESSES

5. Gaussian Processes

Problem 5.1

The probability density of a non degenerate Gaussian vector X is given by:

$$p(x) = \frac{1}{\sqrt{\det(2\pi K)}} \exp\left\{-1/2(x-a)^T K^{-1}(x-a)\right\}$$

where $a \triangleq \mathbb{E}x$ and $K \triangleq \mathbb{E}(x-a)(x-a)^T$. Show that characteristic function $\Phi(\lambda) \triangleq \mathbb{E}e^{i\lambda^T x}$ is given by:

$$\Phi(\lambda) = \exp\left\{i\lambda^T a - \frac{1}{2}\lambda^T K \lambda\right\} \tag{5.1}$$

Degenerate Gaussian vector may be defined by means of (5.1), since it does not involve matrix inversion. Note that constant random variable can be interpreted as Gaussian.

Problem 5.2 (*)

(S.N. Bernshtein)

Let ξ and η be i.i.d. random variables with finite variance and twice differentiable probability density. Show that if $\xi+\eta$ and $\xi-\eta$ are independent, then ξ and η are Gaussian.

Problem 5.3

Let ξ_1, ξ_2 and ξ_3 be independent Gaussian random variables, $\xi_i \sim \mathcal{N}(0,1)$. Show that:

$$\frac{\xi_1 + \xi_2\xi_3}{\sqrt{1+\xi_3^2}} \sim \mathcal{N}(0,1)$$

Problem 5.4

Let ξ be a random variable with continuously differentiable positive probability density

$$p_\xi(x) = \frac{d}{dx}\mathbb{P}\{\xi \le x\}$$

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The *Fisher* information [1] of ξ is defined:

$$I_\xi = \mathbb{E}\left(\frac{\partial \log p_\xi(x)}{\partial x}\Big|_{x\,:=\,\xi}\right)^2 = \int_{x\in\mathbb{R}} \frac{[p'_\xi(x)]^2}{p_\xi(x)}dx$$

Show that:

$$\mathbb{E}\xi^2 \geq 1/I_\xi$$

and equality holds if and only if $p_\xi(x)$ is Gaussian.

Hint: apply Cauchy-Schwarz inequality.

Problem 5.5

Let $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]$ be a Gaussian vector with zero mean. Show that

$$\mathbb{E}\xi_1\xi_2\xi_3\xi_4 = \mathbb{E}\xi_1\xi_2\mathbb{E}\xi_3\xi_4 + \mathbb{E}\xi_1\xi_3\mathbb{E}\xi_2\xi_4 + \mathbb{E}\xi_1\xi_4\mathbb{E}\xi_2\xi_3$$

Hint: recall the connection between the characteristic function of ξ and its moments

Problem 5.6

Let $f(x)$ be a probability density function of a Gaussian variable, i.e:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-a)^2/(2\sigma^2)}$$

Define function:

$$g_n(x_1, ..., x_n) = \Big[\prod_{j=1}^n f(x_j)\Big]\Big[1 + \prod_{k=1}^n (x_k - a)f(x_k)\Big], \quad (x_1, ..., x_n) \in \mathbb{R}^n$$

(a) Show that $g_n(x_1, ..., x_n)$ is a valid probability density function of some random vector $X = (X_1, ..., X_n)$.
(b) Show that any subvector [2] of X is Gaussian.
(c) What conclusion can be made on the basis of this example ?

Problem 5.7

Let $f(x, y, \rho)$ be a two dimensional Gaussian probability density, so that the marginal densities have zero means and unit variances and the correlation coefficient is ρ. Form a new density:

$$g(x, y) = c_1 f(x, y, \rho_1) + c_2 f(x, y, \rho_2)$$

with $c_1 > 0, c_2 > 0, c_1 + c_2 = 1$.

(a) Show that $g(x, y)$ is a valid probability density of some vector $[X, Y]$.
(b) Show that each of the r.v. X and Y is Gaussian.

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[1] Fisher information enters many formulae of mathematical statistics, e.g. lower bounds on estimation errors, etc.

[2] X is not a subvector of itself in this case

(c) Show that c_1, c_2 and ρ_1, ρ_2 can be chosen so that $\mathbb{E}XY = 0$. Are X and Y independent ?
(d) What does this problem demonstrate ?

Problem 5.8 (*)

Consider the process $(X_n)_{n\geq 0}$ generated by a recursion:

$$X_n = \frac{X_{n-1}\xi_n}{\sqrt{X_{n-1}^2 + \xi_n^2}}$$

subject to X_0 - a standard Gaussian random variable. $(\xi_n)_{n\geq 1}$ is a standard Gaussian i.i.d. sequence, independent of X_0.

(1) Prove that X_n is a Gaussian random variable for any $n \geq 0$.
(2) Is $(X_n)_{n\geq 0}$ a Gaussian process ? Prove your answer.
(3) Find recursions for $m_n = \mathbb{E}X_n$ and $V_n = \mathbb{E}(X_n - m_n)^2$.
(4) Does X_n converge ? If yes, to what limit and in what sense? [3]

Note:

To answer (1) and (3) you will need to prove the following:

Lemma 5.1. *Let α and β be a pair of independent Gaussian random variables with zero means and variances σ_α^2 and σ_β^2. Let:*

$$\gamma = \frac{\alpha\beta}{\sqrt{\alpha^2 + \beta^2}}$$

then γ is Gaussian.

This lemma can be proved in several ways and you are encouraged to prove it as you wish. Though you may follow the advice:

(1) Show that in this case the distribution of γ is determined by distribution of $1/\gamma^2$.
(2) Find the characteristic function of $1/\alpha^2$ and $1/\beta^2$
(3) Use the fact that $1/\gamma^2 = 1/\alpha^2 + 1/\beta^2$ and the independence of α and β to find the distribution of $1/\gamma^2$ and hence also of γ.

[3] no need to show convergence with probability 1