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Knowledge Spillover, Learning Incentives and Economic Growth

by

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Abstract: Knowledge spillover implies that the social value of knowledge is higher than its private value and leads to insufficient private investment in human capital. This paper examines implications for economic growth and offers a remedy. An incentive mechanism that implements the socially optimal outcome is offered based on a learning subsidy and flat income or consumption taxes (each levied at a different phase of the growth process). The scheme is self-financed in that the tax proceeds cover exactly the subsidy payments at each point of time.

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1. Introduction

Knowledge spillovers lead to market failure in human capital investment. This paper investigates possible consequences of this market failure for knowledge-based economic growth and offers a remedy. We find that knowledge spillovers can lead to substantial differences in growth patterns under private and social learning regimes. Aiming at a mechanism that implements the socially optimal outcome, we design a simple incentive scheme consisting of flat learning subsidy and taxes. The scheme is self-financed, in that the tax proceeds cover exactly the subsidy payments at each instant of time, so that no lump sum transfers are needed.

The literature on knowledge-based economic growth can be traced back to Arrow's (1962) learning-by-doing model and Shell's (1966, 1967, 1973) treatment of knowledge assets as an additional sector subject to policy decisions – both were early attempts to endogenize Solow's (1956, 1957) technical change process. The recent literature follows Lucas (1988) who assumed that knowledge accumulation (learning) is a time-consuming activity and incorporated external (spillover) effects (see Barro and Sala-i-Martin 2004, Chapter 5). Here we adopt Shell's approach, which treats learning as an income-consuming activity, and incorporate external knowledge effects. Our formulation enables a complete dynamic characterization of the endogenous growth processes and facilitates the design of an optimal learning mechanism.

The dynamic characterization is based on a necessary condition for sustained growth expressed in terms of the marginal productivity of human and physical capital. An economy that satisfies the growth condition needs also sufficient capital-knowledge endowment to realize its growth potential. The optimal growth processes exhibit a turnpike property (Samuelson 1965, Cass 1966), in that they reach a certain

path (the turnpike) as rapidly as possible (in a sense precisely defined in the text) and proceed along it thereafter. Other growth models that behave in this fashion are discussed in Barro and Sala-i-Martin (op cit.); Tsur and Zemel (2002) found a similar growth pattern in a model of growth under resource scarcity.

The proposed incentive scheme consists of a learning subsidy and income or consumption taxes to finance the subsidy expense without distorting household decisions. The learning subsidy is applied at a constant rate along the turnpike and may be implemented also during the transitional phase (the most-rapid-approach to the turnpike) if capital endowment is large enough. The flat income tax (levied at the same rate for both capital and labor income) is used temporarily during the transitional phase to cover the subsidy payments. Once the turnpike has been reached, the distorting nature of income tax rules out its further use and a flat consumption tax is used instead to finance the subsidy. An attractive feature of this mechanism is that its budget is balanced at each instant of time without a resort to lump sum transfers.

Similar properties have been obtained by Rebelo (1991) who studied tax policies in an extended Lucas framework without external effects and identified the conditions under which consumption or income taxes are neutral to growth. Empirical evidence on the growth effects of taxes seems to depend on the formulation and calibration of the specific model considered, as, e.g., in Lucas 1990, Jones et al. 1993, 1997, Pecorino 1994, Stokey and Rebelo 1995, Glomm and Ravikumar 1998, and Judd 1999. These works study a broad set of issues relating to short- and long-run growth effects of fiscal policy under various political and public spending restrictions, but pay little attention to the external effects of knowledge. Yet, there is ample evidence to suggest the existence of substantial social benefits to education that

are not privately captured (Haveman and Wolfe 1984, Acemoglu 1996, Lochner and Moretti 2001, Wolfe and Haveman 2002). In this work we focus on these social effects, and providing private learning incentives that account for them is the primary purpose of our mechanism.

The paper proceeds as follows: The economic environment – the underlying market conditions and decisions made by firms and households – is presented in Section 2; although fairly standard by now, this section lays down the model equations for the analysis that follows and serves to highlight the external effects of human capital and the difference between private and social processes. A complete dynamic characterization of the optimal private and social growth processes is given in Section 3. These processes are compared in Section 4 to evaluate the inefficiency associated with private learning under Cobb-Douglas production technology and iso-elastic consumption preferences. Section 5 describes the learning policy and shows how it achieves the socially optimal outcome. Section 6 concludes and the appendices contain technical derivations.

2. The economy

The economy consists of a large number of identical households and a large number of identical firms. Households own labor, human capital (knowledge) and asset capital (saving). Firms hire these factors to produce a composite good, operating in a competitive environment to maximize profit at each point of time while taking the factor prices as given. Identical rational firms facing the same market conditions will make the same production decisions. A similar remark holds for the households, which determine the accumulation of human and asset capital in order to maximize the present value of a stream of consumption utilities subject to budget constraints. In equilibrium, the factor prices clear the spot labor and capital markets

at each point of time. To focus attention on endogenous growth, population is assumed constant and exogenous technical change is assumed away. The considerations governing the behavior of all agents are now summarized.

Firms: Firm i employs K_i units of physical capital and L_i workers to produce the output $Y_i = F(K_i, A(h)L_i)B(H)$, where h represents individual (intrafirm) worker's level of human capital, $A(h)$ is a labor-augmenting productivity function, $B(H)$ is an inter-firm technology index that depends on aggregate knowledge $H = Lh$, and $L = \Sigma_i L_i$ is aggregate labor. The productivity index B represents the state of technology adopted by the economy up to the present time and incorporates the external effects of knowledge (Lucas, 1988). It is specified as output augmenting but could enter as labor augmenting (as, for example, in Bils and Klenow 2000) without changing the nature of the results.

The production function F is assumed to be linearly homogenous, thus can be expressed as $F(K_i, A(h)L_i) = L_i A(h) f(k_i/A(h))$, where $k_i = K_i/L_i$ is firm i's capital per worker and f is assumed increasing and strictly concave over $(0,\infty)$ with $f(0) = 0$, $f(\infty) = \infty$, $f'(0) = \infty$ and $f'(\infty) = 0$.

At each point of time, firms observe the aggregate stock of human capital, the wage rate w and the capital rental rate r and demand the capital per worker that maximizes profit per worker $A(h) f(k_i/A(h))B(H) - rk_i - w$, obtaining the first order condition

$$f'(k_i/A(h))B(H) = r. \tag{2.1}$$

The labor market clearing wage rate w coresponds to a vanishing profit, yielding, in view of (2.1),

$$[A(h)f(k_i/A(h)) - k_i f'(k_i/A(h))]B(H) = w. \tag{2.2}$$

Multiplying (2.1) by K_i and (2.2) by L_i, adding the results and summing over all firms gives

$$rK + wL = LA(h)f(k/A(h))B(H) = Y,$$

where $Y = \Sigma_i Y_i$ is aggregate output. Dividing through by L, we find

$$rk + w = A(h)f(k/A(h))B(H) \equiv y(k,h,H). \tag{2.3}$$

Equation (2.3) relates household income (on the left-hand side) to the current levels of physical and human capital.

Households: Households decide on the evolution of their asset holdings (saving), human capital and consumption. With identical households, no lending-borrowing takes place and households' assets coincide with the capital available to the firms. Following Shell (1966, 1967, 1973), we assume that learning is an income-consuming activity so that knowledge accumulation is proportional to learning outlays. Households, then, allocate income between consumption (c), saving ($\dot{k}$ = investment in physical capital) and learning ($\dot{h}$ = investment in human capital). In view of (2.3), the representative household budget constraint is $y(k,h,H) = \dot{k} + \dot{h} + c$. At each point of time, the household decides on the fraction α_t of income devoted to learning, such that

$$\dot{h}_t = \alpha_t y(k_t, h_t, H_t). \tag{2.4}$$

The remaining income is allocated between consumption and saving, so that

$$\dot{k}_t = (1 - \alpha_t)y(k_t, h_t, H_t) - c_t. \tag{2.5}$$

Depreciation of both types of capital is ignored for convenience. Implicit in (2.5) is the assumption that investment in physical capital is reversible (i.e., consumption can derive from the stock of capital at no extra cost).

The representative household derives utility from consumption according to an increasing and concave utility function $u(c)$. Time preferences are represented by the positive utility discount rate ρ. A plan is a continuum of consumption-investment decisions (c_t, α_t), $t \geq 0$, generating the value

$$\int_0^\infty u(c_t) e^{-\rho t} dt .$$ (2.6)

A feasible plan satisfies (2.4), (2.5), $h_t \geq 0$, $k_t \geq 0$, $c_t \geq 0$ and $0 \leq \alpha_t \leq 1$ for all $t \geq 0$, given capital-knowledge endowment k_0 and h_0. (The upper bound on α_t entails investing all income in learning. Alternative exogenous bounds can be assumed at no extra complication.) The optimal plan is the feasible plan that maximizes (2.6). We denote by $V(k_0, h_0)$ the value of (2.6) under the optimal plan.

In solving the optimization problem, the household treats aggregate knowledge H as exogenously given when evaluating the marginal effects of changes in his own knowledge level h, even though it is recognized that in equilibrium $H_t = L h_t$. A social planner, on the other hand, would account for these spillover effects of knowledge. The household, in fact, could be made better off by doing the same. Nonetheless, if everyone else accounts for the external effects of learning, an individual household can improve his position by ignoring the contribution of his own learning to the aggregate knowledge stock, which excludes the socially optimal policy from the set of Nash equilibrium policies. We shall refer to a plan that ignores the external effects of knowledge as *private*; a plan that that accounts for these effects is called *social*. In the following section we characterize the optimal private and social plans. The extent of inefficiency of the former is analyzed in Section 4 and Section 5 offers a self-financed learning mechanism that implements the socially optimal outcome.

3. Dynamic characterization of the optimal private and social processes

The extent of learning at each point of time depends on α – the part of income devoted to support learning activities. Since the dynamic equations (2.4)-(2.5) are linear in α and the objective (2.6) is independent of α, the optimal learning policy at each point of time takes one of three distinct regimes: no learning ($\alpha = 0$), maximal learning ($\alpha = 1$), or singular learning (defined below). Under each regime, consumption and saving are optimally adjusted to the chosen learning regime. This classification allows describing the optimal (h,k) process in terms of two characteristic lines defined in the state space.

The first line is defined by the locus of (h,k) states at which the marginal productivity of human capital, $y_h \equiv \partial y(k,h,H)/\partial h$, equals that of physical capital, $y_k \equiv \partial y/\partial k$, implying that no additional gains can be made by reshuffling investment between k and h when they both increase. This line differs between the private and social plans because the marginal productivity of h (y_h) depends on whether H is taken as an exogenous parameter (private) or as $H = Lh$ (social). Let $\eta_A(h) = A'(h)h/A(h)$ and $\eta_B(H) = B'(H)H/B(H)$ denote the elasticities of internal and external knowledge effects, respectively, $x = k/A(h)$ and $z(x) \equiv f(x)/f'(x) - x$. The marginal productivity of human capital is obtained from (2.3):

$$(p) \quad y_h = f'(x)z(x)A'(h)B(H)$$

$$(so) \quad y_h = [f'(x)z(x) + f(x)\eta_B(H)/\eta_A(h)]A'(h)B(H)$$

(3.1)

where "p" stands for "private" and "so" for "socially optimal". Recalling (2.3), we find $y_k = f'(x)B(H)$. The no-arbitrage condition $y_k = y_h$, then, implies

$$(p) \quad z(x) = \frac{1}{A'(h)}$$

$$(3.2)$$

$$(so) \quad z(x) + \frac{\eta_B(Lh)}{\eta_A(h)}(z(x) + x) = \frac{1}{A'(h)}$$

Notice, recalling $f'(x) > 0$ and $f''(x) < 0$, that $z(x)$ is increasing. Solving for the variable $x = k/A(h)$, we see that (3.2p) defines a line in the h-k plane, which we call the *private singular line* and denote by $k_p^S(h)$. The analogous solution of (3.2so) also defines a line in the h-k plane, called the *social singular line* and denoted $k_{so}^S(h)$. We use the notation $k^S(h)$ to represent either $k_p^S(h)$ or $k_{so}^S(h)$ when no confusion arises. It turns out that $k^S(h)$ is the unique locus of (h,k) points along which the singular learning policy is supported (see claim 3 of Appendix A). This property explains the name *singular* attached to this line. The second term of (3.2so), representing the contribution of the external effects, is positive. Since $z(x)$ is increasing, this term implies that $k_p^S(h) > k_{so}^S(h)$. This difference between the two singular lines turns out to be essential to the analysis of the difference between the corresponding growth processes.

To understand the economic significance of the singular line note that above the line, when $k > k^S(h)$, the relation $y_k < y_h$ holds, so that capital is less productive than knowledge hence investment in human capital is more attractive than saving (see Claim 7 in Appendix A). The opposite situation holds below the singular line. Along the singular line the two forms of capital are equally productive at the margin and we expect the decision maker (household for the private plan and the social planner for the social plan) to be indifferent between investing in one or the other. Indeed, we find that if the economy grows indefinitely, the optimal (h,k) process first approaches the singular line at a most-rapid-learning rate: maximal learning ($\alpha = 1$) above the line

and no learning ($\alpha = 0$) below it. Once the singular line has been reached, singular

learning (with $0 < \alpha < 1$) is adopted, adjusting the optimal process to evolve along the

singular line. As this behavior is akin to turnpike models of growth (Samuelson 1965,

Cass, 1966), we refer to the singular line also as the *turnpike*.

The second characteristic line in the (h,k) plane is defined by equating the

marginal productivity of capital y_k with the utility discount rate ρ. Using (2.3), this

condition becomes

$$f'(k/A(h))B(H) = \rho. \tag{3.3}$$

Solving for k, we find

$$k(h) = A(h)f'^{-1}(\rho/B(Lh)). \tag{3.4}$$

The properties of A, f and B ensure that $k(h)$ is increasing. The function $k(h)$

represents the optimal steady state of k when no learning takes place and human

capital is fixed. We thus refer to $k(h)$ as the *steady-state line*. Indeed, if the private or

social knowledge-capital process ever approaches a finite steady state (when learning

is allowed), the steady state must fall on this line (Claim 1 in Appendix A). Notice,

from (3.3) and (2.1), that $r = \rho$ along the steady state line; this equality must hold at

any finite steady state.

We can now state the following property:

Property 3.1: (i) The optimal knowledge and capital processes must either converge

to a steady state on the steady-state line or grow indefinitely along the singular line.

(ii) For a growing economy, the optimal (h,k) processes approach the singular line at a

most-rapid-learning rate, i.e., no learning ($\alpha = 0$) below the line and maximal

learning ($\alpha = 1$) above it, and evolve along it thereafter. $\square$

The property holds for both the private and social plans by considering the private and social singular lines, respectively. Part (ii) of the property establishes the singular line as the turnpike for growing economies. The final (singular) phase follows from Part (i). The initial (most-rapid-approach) phase is understood recalling that only the extreme learning regimes ($\alpha = 0$ or $\alpha = 1$) are allowed away from the singular line. To determine the particular learning regime recall that below the singular line physical capital is more productive at the margin than human capital ($y_k > y_h$) while the reverse relation holds above it. The proof is given in Appendix A.

What are the conditions under which economies grow? As it turns out, these conditions depend on the relative location of the two characteristic lines at large knowledge levels. Two cases are considered, classifying economies into one of two possible types:

(i) The singular line lies above the steady-state line at large h, i.e.,

$$lim_{h \to \infty} [k^S(h) - k(h)] > 0.$$

(ii) The singular line does not exceed the steady-state line at large h, i.e.,

$$lim_{h \to \infty} [k^S(h) - k(h)] \leq 0.$$

Economies that satisfy condition (i) are referred to as *converging* economies and those satisfying condition (ii) are called *potentially growing* economies. Economies of the first type eventually stagnate at a finite steady state, whereas those of the second type have the capacity to grow indefinitely pending "appropriate" capital-knowledge endowment. The type classification depends on the production functions and on the utility discount rate ρ but not on the instantaneous utility $u(c)$, which does not enter the definitions of $k^S(h)$ or $k(h)$. Notice that because the singular line takes on different forms under the private and social plans, the same economy can be of a different type under each of these plans. With the social singular line lying below its private

counterpart, an economy that is potentially growing under the private plan maintains its type under the social plan. However, a private converging economy can obtain either type under the social plan. Explicit examples of these situations are considered in the next section.

The large-h relations that define the two types of economy either extend all the way down to the initial knowledge state (if the characteristic lines never cross), or are reversed at some point (if the lines cross). We consider here only situations where the two lines cross at most once. Multiple crossing introduces some ambiguity regarding the identification of the optimal steady states, but otherwise yields no further insight and is therefore ignored (see Tsur and Zemel, 2001, for a discussion of multiple equilibria in a related context.) Denoting the intersection point by $(\hat{h}, \hat{k})$, we see that the singular line of a converging economy is above the steady-state line for all $h > \hat{h}$ (or for all h if the lines never cross above h_0). For potentially growing economies, the geometrical relation is reversed.

To understand the significance of the relative location of the two lines, we refer to the property that consumption typically decreases above the steady-state line (except under maximal learning where $\alpha = 1$ – see Claim 2). If the optimal (h,k) process of a converging economy were to grow along the singular line from some time onward, it would eventually enter the domain where the singular line lies above the steady-state line and from that time the consumption process would have to decrease indefinitely. But such a policy cannot be optimal, since entering a steady state at any point along such a decreasing consumption path is feasible and yields a higher welfare. This explains the name "converging" attached to this type of economies.

While $y_h = y_k$ holds along the singular line, $y_h > y_k$ holds above it (Claim 4, Appendix A). Thus, it cannot be optimal for a potentially growing economy to settle at a steady state above the singular line, since $y_h > y_k$ implies that the economy would be better off converting some of its physical capital to knowledge (Claim 6). If the knowledge-capital endowment suffices to raise the knowledge stock of a potentially growing economy above $\hat{h}$ (that is, to the domain where the steady-state line lies above the singular line), then the process cannot converge, since the steady state would have to be located above the singular line. The only possibility left for this economy is to grow along the turnpike, explaining the name "potentially growing" attached to this type of economies. Observe that these considerations do not rule out steady states with $h < \hat{h}$ below the singular line. Indeed, potentially growing economies may not realize their growth potential if their capital-knowledge endowment is too small.

Consider a converging economy endowed with a knowledge stock $h_0 < \hat{h}$ so that the geometry of Figure 1 is obtained. Possible state-space evolution paths for this economy corresponding to different capital endowments k_0 are depicted in the figure. The arrows indicate the directions in which the processes evolve over time.

Figure 1

Excluding cases with exceedingly high capital endowment (above $k^1(h_0)$ of Figure 1), the optimal policy of a converging economy is a most-rapid-learning-approach to the singular line ($\alpha_t = 0$ below the singular line and $\alpha_t = 1$ above it), followed by a ride along the singular line to a steady state at the intersection point $(\hat{h}, \hat{k})$. When $k_0 = k^1(h_0)$, the maximal learning policy ($\alpha_t = 1$) drives the (h,k) process directly to $(\hat{h}, \hat{k})$. When $k_0 > k^1$, the optimal plan begins with $\alpha_t = 1$, steering the (h,k)

process above the intersection point $(\hat{h},\hat{k})$ and switching to the no learning policy

$(\alpha_t = 0)$ at some point before reaching the singular line. From that time onward, the

processes converge to a steady state on the steady-state line right below the switching

point. In all cases this economy encourages some learning, increasing knowledge to

$\hat{h}$ or higher. The equilibrium capital stock is also increased relative to the level $k(h_0)$

that would have been obtained without the option to learn and accumulate knowledge.

Time trajectories of learning, knowledge and capital processes are shown in

Figures 2-3 for the cases $k_0 < k^S(h_0)$ and $k^1(h_0) > k_0 > k^S(h_0)$. Notice that learning may

not be initiated at the outset: with small endowment, learning is delayed to allow

capital build up until the turnpike is reached (at $k^S(h_0)$), at which time learning begins

and tuned so as to steer the (h_t, k_t) process along the turnpike (Figure 2). In contrast,

larger endowments call for maximal learning $(\alpha_t = 1)$ immediately (Figure 3). Notice

also that the most-rapid-learning-approach to the singular line can give rise to a non-

monotonic evolution of the capital process k_t (Figure 3).

Figures 2-3

Possible state-space evolution paths for potentially growing economies with

different endowments are depicted in Figures 4-5, with the arrows, again, indicating

the directions in which the processes evolve. The endowment is related, for each

path, to some threshold capital stocks defined in Appendix A.

Figures 4-5

Potentially growing economies blessed with sufficient endowments reach the

turnpike at a most rapid learning approach and grow along it thereafter. Poorer

economies, however, ignore the learning option altogether and converge to a poverty

trap at $k(h_0)$. In some intermediate cases (depicted as the middle trajectory of Figure

4), learning is worthwhile for some period, but being too poor to carry these activities

all the way to the turnpike, the household terminates learning at some point above the turnpike and converges to the steady state on the steady state line below. The complete characterization of the various trajectories is given in Appendix A.

Notice that the human and physical capital endowments are substitutes to some extent. For example, if $h_0 > \hat{h}$ the characteristic lines do not cross in the relevant domain. A potentially growing economy, then, will grow indefinitely with any positive capital endowment. Similarly, a sufficiently large initial capital stock calls for some learning that may be temporary (if the economy eventually settles at a steady state) or permanent (if it grows).

4. Consequences of knowledge spillover

We study the effects of knowledge spillover by comparing the private and social plans for an economy characterized by:

$$f(x) = \theta x^{\beta}, \ 0 < \beta < 1 \text{ (Cobb-Douglas)}; \tag{4.1a}$$

$$A(h) = h^{a}, \ 0 < a < 1; \tag{4.1b}$$

$$B(H) = (H/L)^{b}, \ 0 < b < 1 \tag{4.1c}$$

and the iso-elastic utility

$$u(c) = (c^{1-\sigma} - 1)/(1-\sigma), \ \sigma > 1. \tag{4.1d}$$

Under (4.1b-c), the elasticities $\eta_A(h) = A'(h)h/A(h)$ and $\eta_B(H) = B'(H)H/B(H)$ reduce to the constants a and b, respectively. It will prove useful to introduce the notation

$$\eta = \beta/(1-\beta) \text{ and } \varphi = \theta \beta^{\beta} (1-\beta)^{1-\beta}.$$

Specification (4.1) corresponds to an economy that faces a menu of technologies (developed elsewhere) from which to adopt. Due to setup cost, know-how spillovers and other external effects associated with technology adoption, the technology index B depends on aggregate knowledge (Lucas 1988, Bils and Klenow 2000). The

parameter θ represents social infrastructure, such as corruption level, quality of institutions and property right enforcement (Hall and Jones 1999, De Soto 2000, Easterly 2001, Parente and Prescott 2002).

Under the above specification, the singular lines (equations (3.2)) and the steady-state line (equation (3.4)) specialize to

$(p) \quad k_p^S(h) = \dfrac{\eta}{a} h,$

$(so) \quad k_{so}^S(h) = \eta h$

(4.2)

and

$$k(h) = (\beta\theta / \rho)^{1/(1-\beta)} h^{a+b/(1-\beta)}. \tag{4.3}$$

Recalling the type classification of Section 3, we see that $a + b/(1-\beta) < 1$ implies a converging type under both the private and social plans and rules out sustained growth. We thus assume $a + b/(1-\beta) \geq 1$ and focus on the threshold case

$$a + b/(1-\beta) = 1. \tag{4.4}$$

Under (4.4), the steady-state line (4.3) reduces to the straight line

$$k(h) = (\beta\theta / \rho)^{1/(1-\beta)} h. \tag{4.5}$$

Since the singular and steady state lines are straight lines emanating from the origin, they cannot intersect at positive h values. The analysis of Section 3, then, implies that the economy is converging when the singular line lies above the steady-state line and is potentially growing when

$(p) \quad \rho < a^{1-\beta}\varphi$

$(so) \quad \rho < \varphi$

(4.6)

It is verified that under (4.1) the right-hand side of (4.6) equals the marginal productivity of capital ($\partial y/\partial k$) along the singular line (4.2), which, according to (2.1), determines the interest rate:

$(p)\ r_p = a^{1-\beta}\varphi$

$(so)\ r_{so} = \varphi.$

(4.7)

We see from (4.6)-(4.7) that sustained growth requires that the equilibrium interest rate along the turnpike exceeds impatience.

Since a steady state cannot occur above the singular line (Claim 6), a potentially growing economy in this case (of non-intersecting characteristic lines) will realize its growth potential for any positive capital endowment. Moreover (see Appendix B for a proof):

Property 4.1: Under the specifications (4.1), (4.4) and the growth condition (4.6), the optimal private and social (h,k) processes reach the respective private and social turnpikes (4.2) at a most-rapid-learning rate ($\alpha = 0$ or $\alpha = 1$ while below or above the turnpike, respectively) and grow exponentially along their respective turnpikes at the rate

$(p)\ g_p = \dfrac{a^{1-\beta}\varphi - \rho}{\sigma} = \dfrac{r_p - \rho}{\sigma}$

$(so)\ g_{so} = \dfrac{\varphi - \rho}{\sigma} = \dfrac{r_{so} - \rho}{\sigma}$

(4.8)

The turnpike growth rate is achieved by devoting to learning the constant income fraction

$(p)\ \alpha_p = \dfrac{1 - \rho/(a^{1-\beta}\varphi)}{\sigma}a(1-\beta) = a(1-\beta)\dfrac{r_p - \rho}{\sigma r_p} = a(1-\beta)g_p/r_p$

$(so)\ \alpha_{so} = \dfrac{1 - \rho/\varphi}{\sigma}(1-\beta) = (1-\beta)\dfrac{r_{so} - \rho}{\sigma r_{so}} = (1-\beta)g_{so}/r_{so}$

(4.9)

Saving obtains the income fraction

$$(p) \quad s_p = \dot{k}_p / y_p = \eta \alpha_p / a = \beta g_p / r_p$$

(4.10)

$$(so) \quad s_{so} = \dot{k}_{so} / y_{so} = \eta \alpha_{so} = \beta g_{so} / r_{so}$$

and the residual income fraction

$$(p) \quad c_p / y_p = 1 - \alpha_p - s_p = 1 - (1-b)g_p / r_p$$

(4.11)

$$(so) \quad c_{so} / y_{so} = 1 - \alpha_{so} - s_{so} = 1 - g_{so} / r_{so}$$

is consumed. $\square$

The growth conditions (4.6) ensure that g_p and g_{so} are positive while $\sigma > 1$ implies that α_p lies between 0 and $a(1-\beta)$ and α_{so} lies between 0 and $1-\beta$. The endogenous exponential growth of (4.8) is directly linked to the constant returns to human capital h in the knowledge production function (2.4), implied in equilibrium by (4.4) along the singular line (where k is proportional to h). (Solow 2000 discusses equivalent assumptions in a variety of endogenous growth models.)

From (4.8), we obtain

$$\sigma(g_{so} - g_p) = r_{so} - r_p = \varphi(1 - a^{1-\beta}) > 0,$$

(4.12)

hence (as expected) the private economy grows too slowly. The reason can be traced to insufficient learning on the part of households, as can be seen from

$$\sigma(\alpha_{so} - \alpha_p)/(1-\beta) = 1 - a - (1 - a^{\beta})\rho / r_{so} > 0,$$

(4.13)

which follows from (4.9), (4.7) and the growth condition (4.6). In fact, households also save too little (under the private plan), as can be seen from

$$s_{so} - s_p = \beta \left(\frac{g_{so}}{r_{so}} - \frac{g_p}{r_p} \right) = \frac{\beta \rho(r_{so} - r_p)}{\sigma r_{so} r_p} > 0.$$

(4.14)

This leaves a smaller income fraction for consumption under the socially optimal policy. However, since this fraction is derived from y_{so} which grows faster than its

private counterpart y_p, the overall welfare obtained under the socially optimal plan is larger.

Table 4.1 compares the turnpike values of various economic variables for the private and socially optimal scenarios. The results in Table 4.1 are relevant when both the private and socially optimal plans give rise to growing economies, i.e., when both conditions (4.6) hold. The difference induced by knowledge spillover is even more pronounced when $a^{1-\beta}\varphi < \rho < \varphi$. In this case, the steady-state line $k(h)$ lies between the private and social singular lines, implying that the economy (eventually) stagnates under the private plan but growth exponentially under the social policy.

Table 4.1: Private and social outcomes.

	Private	Social	Disparity
Interest rate	$r_p = a^{1-\beta}\varphi$	$r_{so} = \varphi$	$r_{so} - r_p = \varphi(1 - a^{1-\beta}) > 0$
Growth rate	$g_p = \dfrac{r_p - \rho}{\sigma}$	$g_{so} = \dfrac{r_{so} - \rho}{\sigma}$	$g_{so} - g_p = \dfrac{r_{so} - r_p}{\sigma} > 0$
Learning	$\alpha_p = a(1-\beta)\dfrac{g_p}{r_p}$	$\alpha_{so} = (1-\beta)\dfrac{g_{so}}{r_{so}}$	$\dfrac{\sigma(\alpha_{so} - \alpha_p)}{(1-\beta)} = 1 - a - (1 - a^\beta)\dfrac{\rho}{r_{so}} > 0$
Saving	$s_p = \beta\dfrac{g_p}{r_p}$	$s_{so} = \beta\dfrac{g_{so}}{r_{so}}$	$s_{so} - s_p = \dfrac{\beta\rho(r_{so} - r_p)}{\sigma r_{so}r_p} > 0$
Consumption	$\dfrac{c_p}{y_p} = 1 - (1-b)\dfrac{g_p}{r_p}$	$\dfrac{c_{so}}{y_{so}} = 1 - \dfrac{g_{so}}{r_{so}}$	$\dfrac{c_{so}}{y_{so}} - \dfrac{c_p}{y_p} = \dfrac{-\rho(r_{so} - r_p)}{\sigma r_{so}r_p} - b\dfrac{g_p}{r_p} < 0$

The rightmost column of Table 4.1 presents the growth effects of knowledge spillovers. Whether these effects are indeed appreciable depends on the parameter values, in particular on a and b – the exponents representing the internal and external effects of knowledge. To gain insight on the significance of knowledge spillover effects we evaluate relative turnpike values of the variables of Table 4.1 for representative parameters. Following the literature (e.g., Mankiw et al. 1992) capital share is assumed at $\beta = 1/3$. Regarding the private and social effects of knowledge we

take as a benchmark the assessment that they are of the same order of magnitude

(Wolfe and Haveman 2002). However, in order not to overstate the external effects,

we allow for a larger private contribution by assuming $a/b = 3$. It then follows from

(4.4) that $b = 2/9$ and $a = 2/3$. Moreover, we set $\theta = 0.099$ so as to obtain the private

interest rate of 0.04 (i.e., 4% return to capital along the private singular line). The

corresponding social interest rate (the return to capital along the social singular line)

then equals 5.24 %. Thus

$r_p = 0.04$ and $r_{so} = 0.0524$.

With a utility discount rate of $\rho = 0.03$, we obtain $g_{so}/g_p = 2.24$, $\alpha_{so}/\alpha_p = 2.56$

and $s_{so}/s_p = 1.71$: the turnpike growth rate under the social policy is more than twice

its private counterpart and this faster growth is achieved by higher investment in

human capital (the fraction of income allocated to human capital investment under the

social policy is 2.56 times the corresponding fraction under the private policy). The

corresponding figures for a more forward-looking economy with $\rho = 0.01$ are

$g_{so}/g_p = 1.41$, $\alpha_{so}/\alpha_p = 1.62$ and $s_{so}/s_p = 1.08$: the effects of knowledge spillover on

growth rate and on human capital investment are significant for this economy as well.

The value 0.099 assigned to the social infrastructure parameter θ has been

chosen to give $r_p = 0.04$. It is of interest to find the minimal value of θ that supports

growth. When $\rho = 3\%$, reducing θ below 0.0743 leads to a violation of the growth

condition (4.6p) and transforms a growing private-learning-economy into a

converging (stagnating) type. Reducing θ further below 0.0567 violates (4.6so) and

transforms the social economy into a converging type. For values of θ between

0.0743 and 0.0567, private learning entails stagnation while the social learning policy

supports long run growth.

5. Optimal learning regulation

The failure of the competitive equilibrium to induce appropriate human capital investment calls for policy intervention. In actual practice such an intervention takes different forms, including state-financed schooling and training, and subsidized higher education (see Wolfe and Haveman 2002 for data on public spending on education in a number of countries). In this section we offer a mechanism that implements the socially optimal outcome for the economy specified in Section 4. The mechanism consists of a learning subsidy to encourage human capital investments and linear taxes to cover the subsidy payments. The taxes alter between a flat income tax during the transitional phase (while the knowledge-capital processes approach the turnpike) and a flat consumption (or value added) tax along the singular line. The tax proceeds exactly match the subsidy outlays at each point of time, so no lump sum transfers are needed.

Learning subsidy: For every income unit spent on learning, the household receives a coupon worth q income units that can be used to pay for additional learning only. This policy modifies the human capital accumulation process (2.4) to

$$\dot{h} = \alpha(1+q)y(k,h,H).$$
(5.1)

The value of q that provides the right learning incentives turns out to be $(1-a)/a$. To avoid overinvestment in human capital, that is to ensure that $\alpha(1+q)$ does not exceed unity, the subsidy is given only while $\alpha \le a$. The subsidy policy can be succinctly defined in terms of

$$q = \begin{cases} (1-a)/a & if\ \alpha \le a \\ 0 & otherwise \end{cases}.$$
(5.2)

It is verified in Appendix B that this policy changes households' learning decisions (α_t) in such a way that their private singular line changes from $k_p^S(h)$ to $k_{so}^S(h)$.

Taxes: To balance the mechanism's budget, a flat income tax at a rate m or a flat consumption tax at a rate v is used. These taxes modify the household's saving process (2.5) to

$$\dot{k} = (1 - \alpha - m)y - (1 + v)c. \tag{5.3}$$

The consumption tax v is applied along the turnpike $k_{so}^S(h)$ at the rate $bg_{so}/(r_{so} - g_{so})$:

$$v = \begin{cases} bg_{so}/(r_{so} - g_{so}) & \text{if } k = k_{so}^S(h) \\ 0 & \text{otherwise} \end{cases}. \tag{5.4}$$

Above $k_{so}^S(h)$ an income tax at the rate $1 - a$ replaces the consumption tax, thus

$$m = \begin{cases} 1 - a & \text{if } k > k_{so}^S(h) \\ 0 & \text{otherwise} \end{cases}. \tag{5.5}$$

No subsidy or taxes are used below the singular line.

Efficiency: The tax proceeds exactly match the subsidy payments when

$$q\alpha y = my + vc. \tag{5.6}$$

A *q-v-m* policy that satisfies (5.6) at all times is called feasible. The policy is efficient if it yields the socially optimal (h,k) process. Indeed, we verify in Appendix C that:

Property 5.1: The mechanism defined by (5.1)-(5.5) is feasible and efficient. □

The centerpiece of this mechanism consists of the learning subsidy that modifies the household's singular line from the private $k_p^S(h)$ to the socially optimal $k_{so}^S(h)$. The subsidy induces households to increase the income fraction they devote to

learning from α_p to the regulated fraction $\alpha_r = a\alpha_{so}$, which is still below the socially optimal fraction α_{so}. According to (5.1) and (5.2), however, the subsidy ensures that the fraction α_r implies the correct socially optimal investment in human capital.

Along the turnpike, the consumption tax has no effect on household decisions (Rebelo 1991 showed that this property holds also for an extended Lucas model). Thus, this tax can serve to finance the learning subsidy. Applied at the rate $v=bg_{so}/(r_{so}-g_{so})$, this tax will raise the exact amount needed to cover the subsidy expense. This goal cannot be achieved with the income tax, since its distorting effects divert households away from the social turnpike $k_{so}^S(h)$. Above the turnpike, however, income tax is used temporarily to guide the process towards $k_{so}^S(h)$. At this stage, the income tax serves a dual role: first its distorting effects modify the saving-consumption balance and lead the (h,k) process towards the turnpike at the socially optimal pace. Second, its proceeds match exactly the subsidy payments, so that the consumption tax can be avoided until the turnpike is reached.

Below $k_{so}^S(h)$ (at (h,k) states satisfying $k < k_{so}^S(h)$), the socially optimal policy is to avoid learning and build up capital until the singular line is reached. The regulator, then, has no reason to support learning and the subsidy is set to zero. As a result, the taxes can also be avoided during this phase.

6. Concluding comments

There is a wealth of evidence to suggest that external effects of human capital – social benefits not privately captured – are substantial. This paper investigates growth consequences of these effects and finds them to be significant under a wide range of circumstances. In the extreme case, an economy that sustains long run growth under the social policy would stagnate under the private policy. We offer a

simple mechanism, based on a flat learning subsidy and altering income-consumption taxes, that implements the socially optimal policy. The mechanism is self-financed in that the linear income and consumption taxes, each levied during a different phase of the growth process, match the subsidy payments at each point of time.

Growth failures in our model can happen for two reasons: either the economy fails to satisfy the growth condition, or it lacks resources to realize its growth potential. The former situation is often due to poor social infrastructure, such as corruption, excessive bureaucracy, or insufficient enforcement of property rights. External infusion of capital – a common means of foreign aid for stagnating economies – can jumpstart economic growth for economies that satisfy the growth condition but not for those that fail in this respect (Burnside and Dollar 2000, Easterly 2003). For the latter economies, structural changes must take place in order to escape poverty. A policy that provides the right learning incentives may turn a stagnating economy into a growing one and in general accelerates growth.

Appendix A: Derivation of the optimal plan

We derive here the optimal private plan. The derivation of the socially optimal plan follows the same line and is therefore omitted. The decision problem entails finding c_t and α_t, $t \geq 0$, according to

$$V(k_0, h_0) = Max_{\{c_t, \alpha_t\}} \left\{ \int_0^\infty u(c_t) e^{-\rho t} dt \right\} \tag{A.1}$$

subject to (2.5), (2.6), $k_t \geq 0$, $c_t \geq 0$ and $0 \leq \alpha_t \leq 1$, given the endowments k_0 and h_0. The current-value Hamiltonian is

$$H_t = u(c_t) + \lambda_t[(1-\alpha_t)y(k_t, h_t, H_t) - c_t] + \gamma_t \alpha_t y(k_t, h_t, H_t), \tag{A.2}$$

where λ and γ represent the current-value costate variables of k and h, respectively. Necessary conditions for optimum include

$$u'(c_t) = \lambda_t, \tag{A.3}$$

$$\alpha_t = \begin{cases} 1 & if \ \lambda_t < \gamma_t \\ 0 & if \ \lambda_t > \gamma_t \\ \alpha_t^S & if \ \lambda_t = \gamma_t \end{cases} \tag{A.4}$$

(α_t^S is the singular solution defined below),

$$\dot{\lambda}_t - \rho\lambda_t = -y_k(k_t, h_t, H_t)[(1 - \alpha_t)\lambda_t + \alpha_t\gamma_t], \tag{A.5}$$

$$\dot{\gamma}_t - \rho\gamma_t = -y_h(k_t, h_t, H_t)[(1 - \alpha_t)\lambda_t + \alpha_t\gamma_t] \tag{A.6}$$

and the transversality conditions

(a) $lim_{t\to\infty}\{k_t\lambda_t e^{-\rho t}\} = 0$ and   (b) $lim_{t\to\infty}\{h_t\gamma_t e^{-\rho t}\} = 0.$ (A.7)

Condition (A.4) identifies three possible learning regimes: no learning ($\alpha = 0$),

maximal learning efforts ($\alpha = 1$) and singular learning ($\alpha = \alpha^S$). The optimal plan

consists of selecting among these three regimes at different phases of the planning

horizon. Given the learning regime, only k remains an independent state variable.

The selection among the three learning regimes, thus, reduces the two-state problem

(A.1) into a series of single-state problems.

The steady-state line $k(h)$ is defined by the solution to $f'(k/A(h))B(Lh) = \rho$ (cf.

(3.3)). This equation is implied by the two conditions $\dot{h} = 0$ (i.e., α=0) and $\dot{\lambda} = 0$

that must hold at any optimal steady-state (see (2.4) and (A.5)). Thus, if the optimal

solution ever approaches a steady state, it must fall on this line. Since we refer to this

property below, we state it as:

Claim 1: An optimal steady state (h^*,k^*) must fall on the steady-state line, i.e.

$k^* = k(h^*)$.■

Below the steady-state line (where $k < k(h)$), the strict concavity of f implies $y_k > \rho$.

Thus, using (A.4) and (A.5), $\dot{\lambda} < 0$ below this line. This, together with (A.3) and

$u''(c) < 0$, implies that $\dot{c} > 0$ also holds below the steady-state line. The reverse

relations hold above the line if $\alpha < 1$, yielding

Claim 2: The optimal consumption process increases in time below the steady-state

line under all learning regimes and decreases in time above the steady-state line when

$\alpha_t < 1$. ∎

Implementing the singular plan $\alpha = \alpha^S$ during a time interval is optimal only if the

condition $\lambda_t = \gamma_t$ (see (A.4)) holds during this interval, implying also $\dot{\lambda}_t = \dot{\gamma}_t$. Using

(A.5) and (A.6) we find that these conditions imply $y_k = y_h$, which defines the singular

line (3.2). We obtain the following characterization:

Claim 3: Singular learning can proceed only along the singular line. ∎

Let $\Lambda(k,h) = y_k(k,h,Lh) - y_h(k,h,Lh)$, so that along the singular line $\Lambda(k^S(h),h)$ must

vanish. Moreover, $f'' < 0$ implies $y_{kk}(k,h,Lh) < 0$ and $y_{hk}(k,h,Lh) > 0$, hence

Claim 4: $\Lambda(k,h) > 0$ below the singular line and $\Lambda(k,h) < 0$ above it. ∎

According to (A.4), the optimal learning rate is determined by $\zeta = \gamma - \lambda$: $\alpha = 1$

is optimal when $\zeta > 0$; $\alpha = 0$ is optimal when $\zeta < 0$ and $\alpha = \alpha^S$ is adopted when

$\zeta = \dot{\zeta} = 0$. Using (A.5) and (A.6), one finds

$$\dot{\zeta} = [(1-\alpha)\lambda + \alpha\gamma]\Lambda(k,h) + \rho\zeta . \tag{A.8}$$

Since the shadow prices are positive, we conclude:

Claim 5: (a) If maximal learning ($\zeta > 0$) is optimal below the singular line, then

$\zeta e^{-\rho t} \to \infty$. (b) If no learning ($\zeta < 0$) is optimal above the singular line, then

$\zeta e^{-\rho t} \to -\infty$. ∎

Observe that allowing the faster-than-exponential divergence of ζ in Claim 5 to proceed permanently is inconsistent with the transversality conditions (A.7). Since a steady state involves no learning, Claim 5b implies

Claim 6: A steady state cannot fall above the singular line. ∎

Claim 5 entails restrictions also on the dynamic processes. For example, under maximal learning, physical capital must decrease. If the maximal learning regime is adopted at or below the singular line, the sub-optimal behavior of Claim 5a will be followed permanently. Hence,

Claim 7: Maximal learning can be optimal only above the singular line. ∎

In fact, maximal learning can hold only during a finite period, after which it must be replaced by either no learning (above the singular line) or singular learning (along the singular line).

When no learning takes place, the capital process is monotonic in time because knowledge remains constant and the problem is essentially one-dimensional. Above the singular line, this regime must involve decreasing capital until the singular line is reached, for otherwise the sub-optimal behavior of Claim 5b will be followed permanently. Now, ζ must be negative when the singular line is reached from above under the no-learning regime (α=0). Since no other plan can hold below the singular line (Claims 3 and 7), this capital decreasing, constant-knowledge plan must converge to a steady state on the steady-state line segment below the singular line.

Initiated below the singular line, a no-learning ($\alpha = 0$) process cannot cross it. Neither can it switch to another regime below the singular line (the singular plan holds only on the singular line and Claim 7 precludes maximal learning below the singular line). The only two possibilities left are to converge to a steady state below

the singular line or to reach the singular line (with $\zeta = 0$) and switch to singular learning. We summarize these considerations in

Claim 8: (a) When initiated above the singular line, a no-learning regime continues permanently and the ensuing (h,k) process converges to a steady state on the steady-state line segment below the singular line. (b) When initiated below the singular line, a no-learning regime process either converges to a steady state below the singular line or reaches the singular line and switches to the singular learning regime. ∎

Once the singular learning regime has been initiated on the singular line (with $\dot{\zeta} = \zeta = 0$) we find, using (A.8), that the (h,k) process cannot leave the singular line without violating Claim 7 or 8 (in other words, the singular regime is trapping). In view of Claim 1, the following characterization holds:

Claim 9: The singular regime process either converges to a steady state on the intersection point $(\hat{h}, \hat{k})$ of the steady-state and singular lines or grows indefinitely along the singular line. ∎

To decide between the two options of Claim 9, consider a singular regime process that grows permanently along a singular line segment above the steady-state line. According to Claim 2, this involves a decreasing consumption process. However, the policy of staying at the initial state (diverting to consumption the resources allocated by the singular plan to increase the capital and knowledge stocks), is feasible and yields a higher utility. Therefore, a singular learning regime that drives the (h,k) process permanently along the singular line above the steady-state line cannot be optimal. Of course, a singular plan that drives the (h,k) process along a segment above the steady-state line during a finite period, and upon reaching the intersection

point moves on to a singular segment below the steady-state line cannot be ruled out. These considerations imply

Claim 10: A singular plan cannot be confined to a segment of the singular line that lies above the steady-state line. ■

We apply these results to characterize the optimal processes corresponding to the economy types introduced in Section 3.

*Converging Economies***:** When the characteristic lines cross, a converging economy is characterized by the property that the steady-state line crosses the singular line from above (Figure 1). It follows from Claims 1 and 6 that an optimal steady state must lie on the steady-state line segment with $h \geq \hat{h}$. Suppose $0 < k_0 < k^S(h_0)$. Claim 7 forbids maximal learning, and singular learning can be take place only along the singular line, thus $\alpha = 0$ is initially optimal. Since $k(h_0) > k^S(h_0)$, Claim 8b implies that it is optimal to delay learning and increase capital until k_t reaches $k^S(h_0)$, and proceeds thereafter along the singular line towards the intersection point. According to Claim 10, it cannot be optimal to continue the singular policy past the intersection point (where the singular line lies above the steady-state line). The only steady state allowed on the singular line by Claim 1 is the intersection point. Thus, we deduce from Claim 9 that the optimal (h,k) process must converge to a steady state at the intersection point $(\hat{h}, \hat{k})$.

With capital endowment larger than $k^S(h_0)$, delaying learning is no longer advantageous (Claim 8a) and the optimal policy is to initially set $\alpha = 1$, increasing knowledge and decreasing capital until the (h,k) process reaches the singular line at some time. From that time on, α is reduced to the singular value, and the process continues along the singular line to the steady state $(\hat{h}, \hat{k})$.

Evidently, the higher the initial endowment k_0, the higher is the point at which the singular line is reached. In fact, there exists some threshold initial stock $k^1(h_0) > k^S(h_0)$ such that the (h,k) process initiated from $(h_0,k^1(h_0))$ with $\alpha = 1$ meets the singular line exactly at $(\hat{h}, \hat{k})$ (see Figure 1). To see this, we solve the dynamic equations for (k,h,λ) backwards in time with $\alpha = 1$, using the reversed time $\tau = -t$ and the initial values $k_{\tau=0} = \hat{k}$, $h_{\tau=0} = \hat{h}$ and $\lambda_{\tau=0} = u'(\hat{c})$ where $\hat{c} = y(\hat{k}, \hat{h}, L\hat{h})$ is the equilibrium consumption rate holding at $(\hat{h}, \hat{k})$. The threshold stock $k^1(h_0)$ is determined from the solution as the state k_τ corresponding to the reversed time τ when $h_\tau = h_0$. Using Claim 4 and the time-reversed version of (A.8) initiated with $\zeta_{\tau=0} = 0$, it is verified that $\zeta_\tau = -\int_0^\tau \Lambda(k_s, h_s)\lambda_s e^{\rho(s-\tau)}ds > 0$ along the solution and maximal learning is indeed optimal all the way back to $(h_0,k^1(h_0))$.

When $k_0 > k^1(h_0)$, the maximal learning plan brings the process to a point $(\hat{h}, k)$ above the singular line. In such cases, this plan continues to higher knowledge stocks, but it cannot cross the singular line (Claim 7) or meet it above the steady-state line (Claim 10). At some point above the singular line the variable ζ vanishes and learning ceases abruptly, leading to a k-decreasing process towards a steady state on the steady-state line segment below the singular line. Thus, $(\hat{h}, \hat{k})$ is the optimal steady state whenever $k_0 \leq k^1(h_0)$, while larger endowments imply higher asymptotic knowledge and capital stocks.

When the steady-state line lies below the singular line for all $h > h_0$, no point along it can be ruled out as a steady state. Of course, optimal trajectories that end by a singular approach to the intersection point are not feasible in this case, but otherwise the characterization above is not affected.

Potentially growing Economies: Here the steady-state line crosses the singular

line from below (Figures 4-5). Claims 1 and 6 restrict optimal steady states to lie on

the steady-state line segment between h_0 and $\hat{h}$. In contrast to the previous,

converging case, Claim 10 forbids the optimal process to converge to the intersection

point $(\hat{h}, \hat{k})$ along the singular line. However, unbounded growth along the singular

line cannot be ruled out. The dynamic behavior, then, depends on two critical capital

stocks defined by the following properties: $k^2(h_0)$ is the maximum endowment for

which it is optimal to avoid learning altogether and approach the steady state

$(h_0, k(h_0))$. (If the endowment $k_0 = k(h_0)$ implies approaching the singular line, set

$k^2(h_0) = 0$.) Obviously, for all $0 < k_0 < k^2(h_0)$ it is optimal to avoid learning and

converge to $(h_0, k(h_0))$. $k^3(h_0)$ is the minimum endowment in excess of $k^S(h_0)$ for

which eventual growth along the singular line is optimal.

If the endowment $k_0 = k^S(h_0)$ implies unbounded singular growth, set $k^3(h_0) =$

$k^S(h_0)$. Otherwise, to find $k^3(h_0)$ note, using Claims 1 and 6, that there must exist a

minimal knowledge level $h_0 < h_m \leq \hat{h}$ such that initiated from the state $(h_m, k^S(h_m))$ on

the singular line, the optimal process follows the singular plan of unbounded growth.

The critical level $k^3(h_0)$ is obtained by solving the dynamic equations for h, k and λ

backwards in time with $\alpha = 1$, using the initial values $(h_m, k^S(h_m))$ and the initial value

of λ determined by the condition that the constant knowledge plan that drives the

system from $(h_m, k^S(h_m))$ to a steady state at $(h_m, k(h_m))$ is also optimal. The critical

level $k^3(h_0)$ corresponds to the (reversed) time when the state $h = h_0$ is reached.

Evidently, $k^3(h_0) \geq k^2(h_0)$, and for all $k_0 > k^3(h_0)$ it is optimal to initially learn at the

maximal rate and drive the process to the singular line and then switch to unbounded

growth under the singular plan.

To characterize the behavior for intermediate endowments with $k^2(h_0)<k_0<k^3(h_0)$ we distinguish between two cases: (i) $k^2(h_0) \geq k^S(h_0)$ (Figure 4) and (ii) $k^2(h_0)<k^S(h_0)$ (Figure 5). In case (i) a maximal learning rate is initially adopted. Learning is then ceased upon the vanishing of ζ at some point above the singular line, and the process crosses the singular line towards a steady state on the steady-state line below (see the intermediate trajectory of Figure 4). Case (ii) implies $k^3(h_0) = k^S(h_0)$ because any point on the singular line gives rise to a growing singular plan. Delayed learning leads the process to $(h_0, k^S(h_0))$. Once the singular line is reached, the singular plan of unbounded growth takes over (see the intermediate trajectory of Figure 5).

If the characteristic lines never cross, Claims 1 and 6 forbid the existence of any steady state, hence the economy must grow permanently along the singular line. When $k_0 < k^S(h_0)$ the no learning regime ($\alpha = 0$) is invoked, increasing capital until $k^S(h_0)$ is reached (Claim 8b), at which time the singular learning regime is adopted to steer the (h,k) process along the singular line. In contrast, when $k_0 > k^S(h_0)$ the no-learning regime is suboptimal (Claim 8a) and maximal learning ($\alpha = 1$) is employed until the singular line is reached and the singular plan takes over. This behavior is described as a most-rapid-learning-approach to the turnpike.

Appendix B: Turnpike growth processes (Properties 4.1 & 5.1)

In Appendix A we establish the basic policy rule for growing economies: reach the singular line at a most-rapid-learning rate and proceed along it thereafter. This policy rule applies to both the private and social plans, which differ only in the location of the corresponding singular line. The private plan under the *q-m-v* mechanism is called the regulated plan. This plan abides by the same policy rule, with its own singular line that depends on the regulation parameters *q*, *m* and *v*. Here we derive the optimal growth processes along the singular lines of the private (*p*),

social (*so*) and regulated (*r*) plans assuming the production technology and utility specified in (4.1) and the growth condition (4.6). As it turns out, the optimal *q-m-v* values of the regulated plan differ between the transitional phase (i.e., during the most-rapid-learning-approach to the singular line) and the equilibrium (singular) phase. In this appendix we analyze the equilibrium phase; the transitional phase is considered in Appendix C.

The derivation is similar for the three plans and we use a generic formulation incorporating all of them as special cases. Thus, we extend (2.4) and (2.5) to account for the regulation mechanism,

(a) $\dot{h} = \alpha(1+q_j)y, \quad j \in \{p,so,r\}$

(b) $\dot{k} = (1-\alpha-m_j)y - (1+v_j)c, \quad j \in \{p,so,r\}$

(B.1)

where q_j, m_j and v_j may differ from zero only for the regulated plan, i.e.,

$$q_j = \begin{cases} 0 & j=p \\ 0 & j=so \\ q & j=r \end{cases} \qquad m_j = \begin{cases} 0 & j=p \\ 0 & j=so \\ m & j=r \end{cases} \qquad v_j = \begin{cases} 0 & j=p \\ 0 & j=so \\ v & j=r \end{cases},$$

(B.2)

and the optimal values of *q*, *m* and *v* are derived below. Unless otherwise indicated, the use of the subscript *j* implies $j \in \{p,so,r\}$.

The specifications in (4.1) give

$$y = \theta h^{a(1-\beta)} k^{\beta} (H/L)^b,$$

(B.3)

hence $y_k = \beta y/k$ for all plans, $y_h = a(1-\beta)y/h$ for the private and regulated plans, in which *H* is taken as a given parameter, and $y_h = [a(1-\beta)+b]y/h = (1-\beta)y/h$ for the social plan (that accounts explicitly for the relation $H/L = h$), where the rightmost equality follows from (4.4).

Following Appendix A, we find that the condition for singular learning generalizes to $y_k = (1+q_j)y_h$, yielding linear singular lines

$$k_j^S(h) = \chi_j h, \tag{B.4}$$

with the slopes

$$\chi_j = \eta /[a(1+q_j)],\ j \in \{p,r\}\ \text{and}\ \chi_{so} = \eta /(1+q_{so}) \tag{B.5}$$

Recalling that $q_{so} = 0$, we see that the subsidy rate leading to $\chi_r = \chi_{so} = \eta$ is

$$q_r = q = (1-a)/a \tag{B.6}$$

as specified in (5.2). Under (B.6), the social and regulated singular lines coincide.

Along the singular line (B.4), the production function (B.3) reduces to

$$y_j^S = \theta \chi_j^\beta h^{1-b}(H/L)^b \tag{B.7}$$

and the marginal knowledge productivity is $\partial y_j^S / \partial h \equiv y_{j,h}^S = \theta \chi_j^\beta (1-b)$ for $j \in \{p,r\}$

and $\partial y_{so}^S / \partial h \equiv y_{so,h}^S = \theta \chi_{so}^\beta$ (only the social planner accounts for the external effects).

Thus, we rewrite (B.7) as

$$y_j^S = Z_j y_{j,h}^S h \tag{B.8}$$

where the coefficient

$$Z_j = 1/(1-b),\ j \in \{p,r\};\qquad Z_{so} = 1 \tag{B.9}$$

accounts for the neglect of the external effects in the evaluation of the marginal productivity of knowledge along the private and regulated singular lines.

It is expedient to express the marginal knowledge productivity as

$$y_{j,h}^S = r_j W_j /(1+q_j) \tag{B.10}$$

where r_j is the interest rate along the corresponding singular lines (see 4.7):

$$r_j = \varphi, \ j \in \{so, r\}, \quad \text{and} \quad r_p = a^{1-\beta}\varphi \tag{B.11}$$

and using (4.4), (B.5)-(B.9) (and some straightforward algebraic manipulations)

$$W_j = 1 + \chi_j(1 + q_j). \tag{B.12}$$

In the following discussion we suppress, for brevity, the index j from the dynamic variables (α, c, y etc.) but continue to use it to distinguish among the constant parameters of the three problems. Inserting (B.4) into (B.1) gives

$\chi_j\alpha(1 + q_j)y^S = (1 - \alpha - m_j)y^S - (1 + v_j)c$, which can be reduced, using (B.12), to obtain the consumption fraction

$$c = \frac{1 - m_j - \alpha W_j}{1 + v_j} y^S. \tag{B.13}$$

In view of (B.13), the singular plans are determined by

$$V_j^S(h_0) = Max_{\{\alpha\}}\left\{\int_0^\infty u[\frac{1 - m_j - \alpha W_j}{1 + v_j}y^S]e^{-\rho t}dt\right\} \tag{B.14}$$

subject to (B.1a), $h_t \geq 0$ and $0 \leq \alpha_t \leq (1 - m_j)/W_j$, where h_0 is reset to the knowledge state at which the singular plan begins. (Similarly, we reset the time at which the singular process starts to $t = 0$.)

The current-value Hamiltonian for this problem is

$$H = u[\frac{1 - m_j - \alpha W_j}{1 + v_j}y^S] + \gamma\alpha(1 + q_j)y^S. \tag{B.15}$$

The necessary conditions include

$$u'(c) = \gamma(1 + q_j)(1 + v_j)/W_j \tag{B.16}$$

and, using (B.10) and (B.16),

$$\dot{\gamma} = \rho\gamma - \partial H/\partial h = \gamma[\rho - r_j(1 - m_j)] \equiv -\gamma\Phi_j, \tag{B.17}$$

where

$$\Phi_j = r_j(1 - m_j) - \rho \; . \tag{B.18}$$

From (B.17), $\gamma = \gamma_0 \, exp(-\Phi_j t)$ and (B.16) implies $\dot{c}/c = \Phi_j u'(c)/[-u''(c)c]$ which for

the isoelastic utility $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ reduces to

$$\dot{c}/c \equiv g_j = \Phi_j/\sigma. \tag{B.19}$$

According to (B.11) $r_r = r_{so} = \varphi$. Since $m_{so} = 0$, we see from (B.18) and (B.19)

that any positive value of the income tax m_r distorts the regulated growth process

relative to its desired (socially optimal) rate. Thus, to obtain the same growth rates

for the regulated and social plans the regulator must set

$$m_r = m = 0 \tag{B.20}$$

along the singular line. We suppress, therefore, all reference to the income tax for the

rest of Appendix B. We also note that the turnpike growth rate is independent of the

consumption tax v_j, suggesting this tax as the appropriate tool to fund the learning

subsidy along the singular line.

Condition (4.6) and $\sigma > 1$ ensure that consumption grows exponentially at the rate

$0 < g_j < r_j$. It turns out that income, capital and knowledge are all proportional to

consumption along the singular line and thus grow at the same rate. This is so

because the optimal learning fraction α_t is constant along the singular line. To verify

this, compare (B.1) with (B.8) and (B.10) to find $\dot{h} = \alpha Z_j r_j W_j h$. Using (B.8), (B.10)

and (B.13) we write the consumption rate as $c = h(1 - \alpha W_j)Z_j r_j W_j/[(1+v_j)(1+q_j)]$. Thus,

$$g_j = \dot{c}/c = \dot{h}/h - \dot{\alpha}W_j/(1-\alpha W_j) = Z_j r_j \alpha W_j - \dot{\alpha}W_j/(1-\alpha W_j) \text{, yielding}$$

$$W_j \dot{\alpha} = g_j(1 - W_j \alpha)(\Omega_j W_j \alpha - 1), \tag{B.21}$$

where $\Omega_j = Z_j r_j / g_j > 1$.

Integrating (B.21) gives $(\Omega_j W_j \alpha - 1)/(1 - W_j \alpha) = \psi_j exp[g_j(\Omega_j - 1)t]$ or

$$W_j \alpha = \frac{1 + \psi_j exp[g_j(\Omega_j - 1)t]}{\Omega_j + \psi_j exp[g_j(\Omega_j - 1)t]}.$$ To set the integration constant ψ_j, note that with

$\Omega_j > 1$ any non vanishing value of ψ_j that avoids divergence at finite time implies that

$W_j \alpha$ converges to unity in the long run with $1 - W_j \alpha \approx exp[-g_j(\Omega_j - 1)t]$ (the notation

$a_t \approx b_t$ signifies that the ratio a_t/b_t approaches a constant as $t \to \infty$). It follows that

$h = c(1 + v_j)(1 + q_j)/[Z_j r_j W_j(1 - W_j \alpha)] \approx exp[g_j \Omega_j t] = exp[Z_j r_j t]$, hence

$h \gamma exp(-\rho t) \approx exp[(Z_j r_j - \Phi_j - \rho)t] = exp[r_j(Z_j - 1)t]$ (cf. (B.18)). Since $Z_j \geq 1$ (see

B.9), the exponent on the right-hand side does not approach zero at large t, violating

the transversality condition (A.7b). Thus, the optimal integration constant ψ_j must

vanish for all plans, reducing the optimal α to the constant $\alpha_j = 1/(\Omega_j W_j)$, which

ensures that $1 - W_j \alpha_j = (r_j Z_j - g_j)/r_j Z_j > 0$.

Using the explicit expressions above for each problem, and recalling that the

growth and interest rates are equal along the social and regulated singular lines, we

find

$$\alpha_p = (1 - \beta)ag_p/r_p, \quad \alpha_{so} = (1 - \beta)g_{so}/r_{so} \quad \text{and} \quad \alpha_r = (1 - \beta)ag_{so}/r_{so} \tag{B.22}$$

verifying (4.9). Note that $\alpha_r = a\alpha_{so} < \alpha_{so}$. Nevertheless, the subsidy policy implies

that regulated learning is enhanced by the factor $1 + q_r = 1/a$, hence proceeds at the

socially optimal rate.

It remains to determine the consumption tax rate $v_r = v$ along the singular line.

This turns out to be a simple task because v affects neither the growth rate nor the

learning fraction along the singular line and can be adjusted to cover the subsidy

payments. Noting (B.13), we see that $q_r \alpha_r y^S = v_r c = v_r(1 - \alpha_r W_r)y^S/(1 + v_r)$, hence

$v_r = \alpha_r q_r / (1 - \alpha_r W_r - \alpha_r q_r)$. From (4.4), (B.6), (B.9) and (B.22) we find that

$\alpha_r W_r = (1-b) g_{so}/r_{so}$ and $\alpha_r q_r = b g_{so}/r_{so}$, yielding

$$v_r = v = b g_{so} / (r_{so} - g_{so}),$$ (B.23)

as specified in (5.4). With v given by (B.23), the consumption/income ratio for the

regulated economy is $c_r / y_r^S = \alpha_r q_r / v_r = 1 - g_{so}/r_{so}$, which is the same as the socially

optimal ratio (see B13). It follows that the regulated and socially optimal

consumption (and the utilities derived thereof) coincide at all times.

Appendix C: The transitional phase of the regulated plan (Property 5.1 cont.)

In Appendix B we analyze the equilibrium growth phase (along the singular

line), setting the subsidy rate q so that the social and regulated singular lines coincide

and imposing the consumption tax rate v such that the tax proceeds just cover the

subsidy payments at each point of time. Here we consider the transitional phase

(away from the singular line) and show that the regulated plan coincides with the

socially optimal policy of a most rapid learning approach to the singular line. The

reasoning follows closely the arguments of Appendix A.

Under (B.6) the r and so singular lines are the same $k_r^S(h) = k_{so}^S(h) = \eta h$ and lie

below their p counterpart $k_p^S(h) = (\eta/a)h$, while the growth condition (4.6) ensures

that the (common) steady-state line lies above all of them. For $k < \eta h$, the socially

optimal policy is to avoid leaning and accumulate capital until the singular line is

reached (Appendix A). The regulator, then, has no reason to support leaning, hence

neither subsidy nor taxes are used below the social-regulated singular line. Without

subsidy or taxes, the conditions of the p problem, which admits no learning below its

own singular line, apply. Thus, the optimal r policy is to set $\alpha_t = 0$ and increase

capital until $k_r^S(h)$ is reached (any other choice entails the contradictions that motivate Claims 3 and 7 of Appendix A).

Consider the optimal learning policy for $k > \eta h$. Equations (A.5) and (A.6) for the regulated process are modified to

$$\dot{\lambda}_t - \rho\lambda_t = -y_k[(1-\alpha_t - m)\lambda_t + \alpha_t(1+q)\gamma_t] \tag{C.1}$$

and

$$\dot{\gamma}_t - \rho\gamma_t = -y_h[(1-\alpha_t - m)\lambda_t + \alpha_t(1+q)\gamma_t]. \tag{C.2}$$

To rule out the no learning policy above the singular line, note that the choice $\alpha_t = 0$ is optimal only when the variable $\zeta = (1+q)\gamma - \lambda$ takes negative values. However, when $\alpha_t = 0$ we find from (C.1) and (C.2)

$$\dot{\zeta} = -\lambda(1-m)[(1+q)y_h - y_k] + \rho\zeta. \tag{C.3}$$

Recalling that $y_k = \beta y/k$ and $y_h = a(1-\beta)y/k$ for the r problem (see B.3), the term inside the square brackets of (C.3) must be positive when $k > \eta h$. Thus, for any income tax rate below unity ($m<1$), a negative value of ζ entails the divergence of $\zeta e^{-\rho t}$ (as in Claim 5b), violating the transversality conditions (A.7). It follows that maximal learning $\alpha_t = 1/(1+q) = a$ is optimal above $k_r^S(h)$. Together with the result that $\alpha_t = 0$ below the singular line, we find that, away from the singular line, the optimal regulated plan is a most-rapid-learning-approach to the singular line from any initial state.

To cover the subsidy cost above the singular line, the tax proceeds must satisfy (5.6) or $vc + my = \alpha q y = (1-a)y$. However, away from the singular line the optimal consumption/income ratio is not necessarily constant hence the only way to balance the budget with constant tax rates is to set $v = 0$ and $m = 1-a$, in accordance with (5.4) and (5.5).

The most-rapid-learning-approach to $k_r^S(h) = k_{so}^S(h)$ implies that once the singular line is reached, the optimal r process must remain on it, following the *so* growth pattern established in Appendix B and yielding the *so* value. For r processes initiated above the singular line, the subsidy and tax rates determined above reduce (B.1) to $\dot{h} = y$ and $\dot{k} = -c$ which agree with the *so* equations of motion under the maximal learning policy of $\alpha_{so} = 1$. Given the learning policy, external effects do not enter the consumption-saving tradeoffs, hence the r and *so* processes must coincide all the way to the singular line. Similar considerations apply to processes initiated below the singular line, which avoid learning and increase capital at the socially optimal rate.

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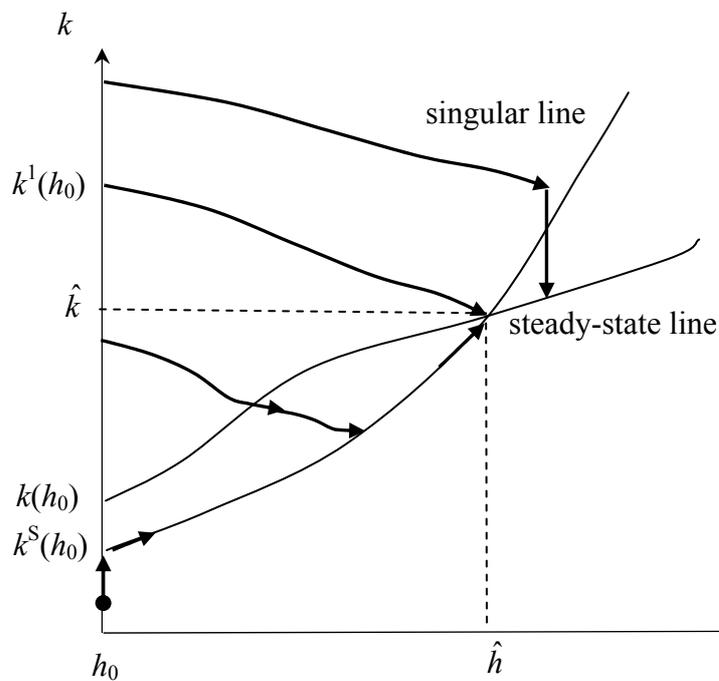
Figure 1: Knowledge-capital processes for a converging economy. Arrows indicate direction of evolution. Economies with capital endowments at or below $k^1(h_0)$ approach the singular line and proceed along it to a steady state at the intersection point $(\hat{h}, \hat{k})$.
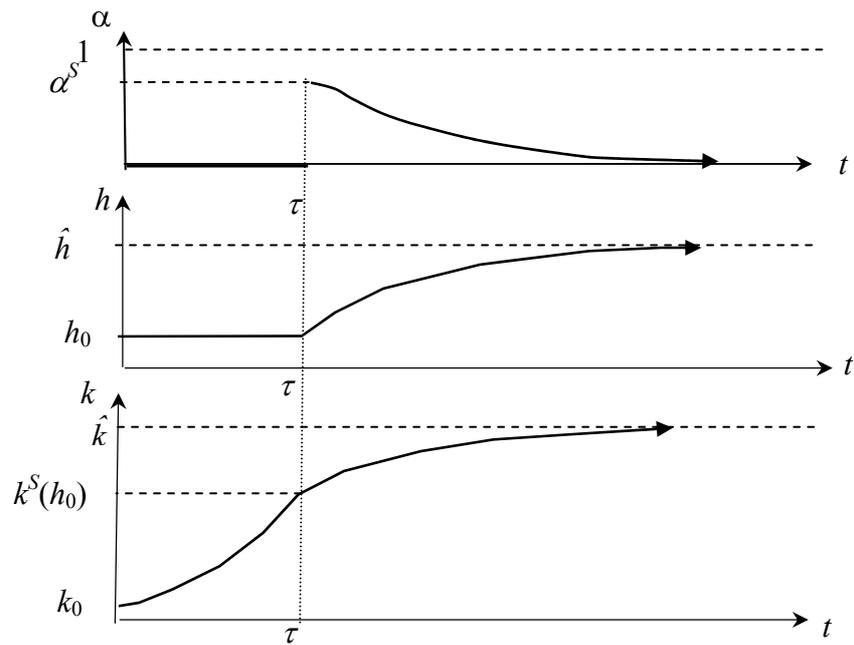
Figure 2: Time profiles of learning fraction (top), knowledge (center) and capital (bottom) processes for a converging economy when $k_0 < k^S(h_0)$. The h and k processes reach the singular line at time τ.
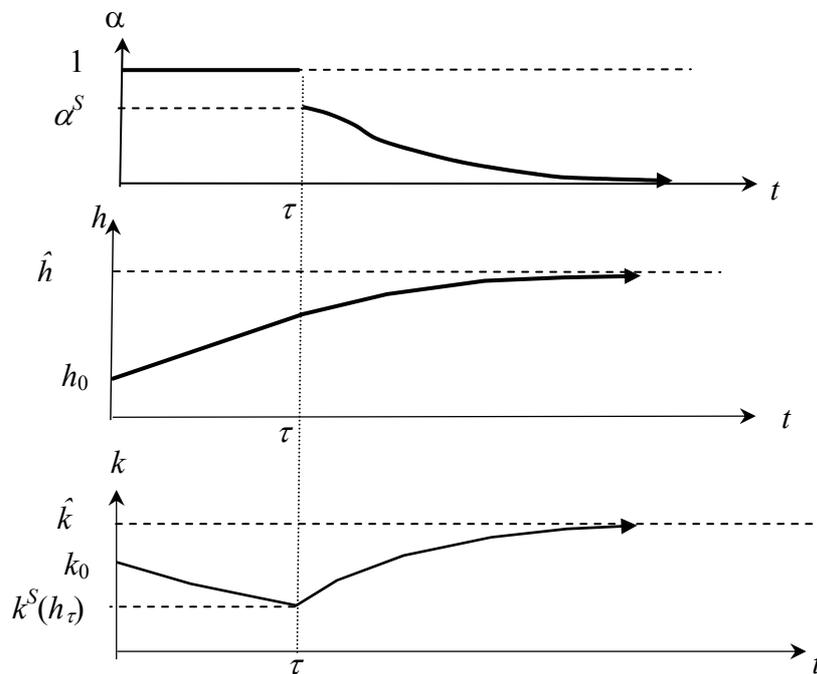
Figure 3: Time profiles of learning fraction (top), knowledge (center) and capital (bottom) processes for a converging economy when $k^1(h_0) > k_0 > k^S(h_0)$. The h and k processes reach the singular line at time τ.
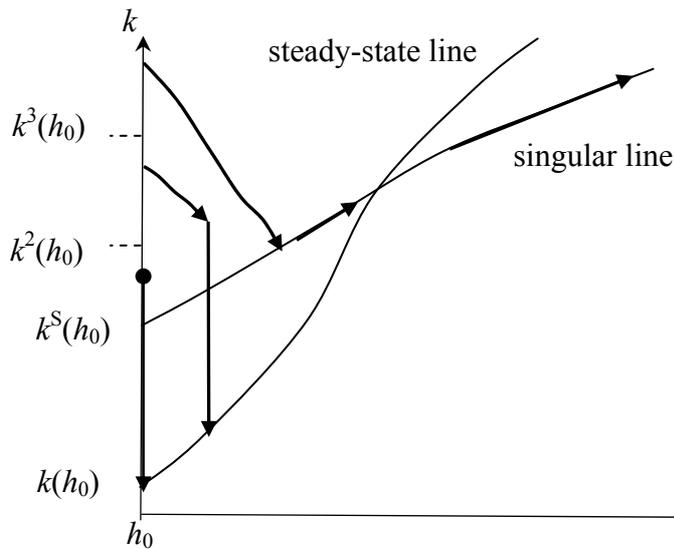
Figure 4: Knowledge-capital processes for a potentially growing economy when $k^2(h_0) > k^S(h_0)$. Arrows indicate the direction of process evolution. When capital endowment is below $k^2(h_0)$, no learning takes place. When capital endowment is between $k^2(h_0)$ and $k^3(h_0)$, maximal learning ($\alpha=1$) takes place initially, then learning ceases and the economy converge to a steady state on the steady-state line. When $k_0 > k^3(h_0)$, the economy approaches the singular line at a most-rapid-learning rate and grows along it thereafter.
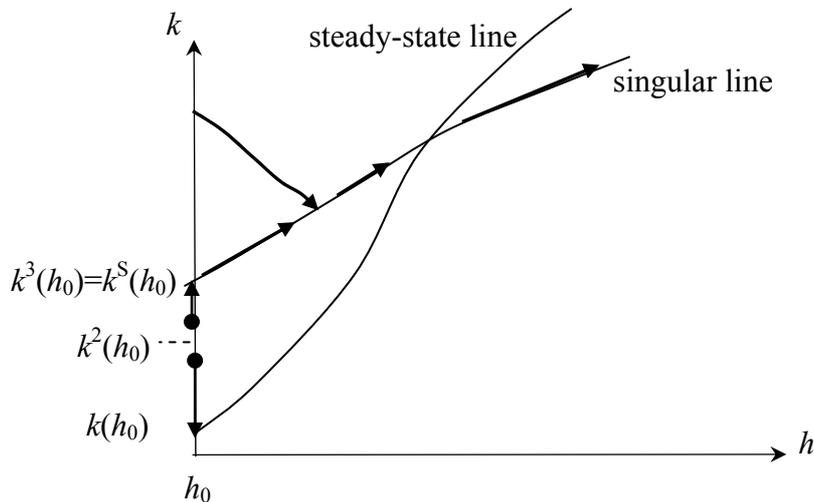
Figure 5: Knowledge-capital processes for a potentially growing economy when $k^2(h_0) < k^S(h_0)$. Arrows indicate the direction of evolution. When capital endowment is below $k^2(h_0)$ no learning takes place and the economy converges to $k(h_0)$. When $k_0 > k^2(h_0)$, the economy approaches the singular line at a most-rapid-learning rate and grows along it thereafter.

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