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Water pricing

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Abstract

The water prices that implement the optimal water policy are derived. These prices contain the supply cost components and two shadow price terms: one reflecting the *in situ* value of natural water and the other representing the scarcity of recycled water. The former accounts for the scarcity, extraction cost and instream value of natural water, and has a pronounced effect on the onset and extent of desalination along the optimal policy. The latter accounts for the scarcity of recycled water, stemming from the limit imposed on its supply by the sewage discharge, and acts as a tax on users of recycled water and as a subsidy for domestic and industrial users that contribute to the supply of recycled water (via the sewage they discharge). Special attention is given to implications of the public good role of environmental water allocation. An example based on Israel's water economy is presented.

Keywords: Water economy, water regulation, water pricing.

JEL classification: C61, Q25, Q28

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1 Introduction

A water price represents the unit value of water of specific quality at a certain point of time, hence varies across quality grades and over time. *Optimal* prices reflect prices along the optimal water policy. In this work I derive the optimal water prices and study their role in water regulation. The analysis is carried out within a comprehensive water economy framework that can accommodate a wide range of real world situations.

Why should we be concerned with calculating water prices? Can't we just rely on water markets to provide this information? The main problem with water markets is that they are far from ubiquitous and, when exist, prone to fail for various reasons, including (*i*) common pool externality (when pumping/diverting water from a shared aquifer, reservoir or stream flow), (*ii*) returns to scale associated with the water infrastructure (which constitutes a considerable share of the cost of water allocation and is shared by many users), (*iii*) supply uncertainty (due to stochastic precipitation), and (*iv*) dependence on water ownership rights, allocation rules and norms. Moreover, water markets take a long time to form and operate properly in any given circumstance, hence rarely exist in young and/or rapidly changing water economies. Absent properly operating water markets, water allocation must be regulated and such regulation is based in one way or another on water prices. In this work I derive the water prices that implement the optimal water policy, specified in the context of the representative water economy framework studied by Tsur (2009) and Tsur and Zemel (2018).

While annual supplies of natural water are on average constant (with a possible moderate long run trend due to climate change), population growth implies that the per capita supply of natural water, measured in cubic meter per person per year (CMpy), declines over time. Dinar and Tsur (2015) estimate that by 2050, roughly 6 billion people in 80 countries are expected to experience water scarcity (below 1000 CMpy) of which 3 billion will suffer from absolute scarcity (below 500 CMpy). A proper management of water resources, thus, becomes a critical policy issue in an increasing number of regions.

The analysis is carried out in the context of a representative (prototypical) water economy that can easily be modified to accommodate many real world situations. A water economy consists of water sources, water users (sectors), the physical capital (infrastructure) that allows allocating water from sources to users, and the institutional setting specifying property rights and feasible allocation rules. Within a given institutional setting, a water policy determines the allocation of water from each source to each sector and the investment in capital infrastructure needed to carry out the water allocation, at each point of time. The purpose of water management is to implement the feasible water policy deemed optimal, based on well-defined and agreed upon criterion.

For such a comprehensive water economy framework, just formulating the optimal policy (before any consideration of implementation) becomes a formidable task, as it entails solving a multi-state intertemporal optimization problem that rapidly becomes analytically intractable. Tsur and Zemel (2018) greatly simplified the derivation of the optimal policy by showing that it evolves along two stages: a *most-rapid-approach* (MRAP) stage followed by a *turnpike* stage.[1] My aim in this work is to derive the water prices that implement this optimal policy. Although water economies vary widely in sources (due to climatic and hydrological variability) and in sectors (due to how populous and developed the economy is), as well as with respect to the institutional setting, the principles underlying optimal policies are universal.

Tsur (2009) formulated a water economy similar to the one considered herein and an-

[1]The term "turnpike" was coined by Dorfman et al. (1958) to represent a common economic situation: "...if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end. The best intermediate capital configuration is one which will grow most rapidly, even if it is not the desired one, it is temporarily optimal" (p. 311). The term "most rapid approach" (MRAP) was coined by Spence and Starrett (1975), who considered an MRAP to a constant target. Here the MRAP is to a moving (turnpike) target (example of an MRAP to a moving target can be found in Tsur and Zemel 2000).

alyzed the optimal steady state policy. Tsur and Zemel (2018) extended the analysis by characterizing the full dynamics of the optimal policy and showed that it evolves along the two aforementioned (MRAP and turnpike) stages. The MRAP stage is concerned with constructing the water infrastructure and is typically short (its duration is inversely related to the investment budget during construction). As soon as the water capital stocks (infrastructure) reach their (well-defined) turnpikes, the optimal policy enters the turnpike stage and evolves along the smooth turnpike trajectories thereafter. Tsur and Zemel (2018) also showed that the turnpike processes eventually approach a steady state (the same steady state considered in Tsur 2009). The present effort is concerned with the implementation of the optimal policy via water pricing.

The water economy is formulated in the next section and the optimal policy (derived in Tsur and Zemel 2018) is summarized in Section 3. The optimal pricing policy, i.e., the water prices that implement the optimal policy, is derived in Section 4 and shown to admit cost recovery (i.e., the proceeds it raises cover the variable and capital supply costs). Special attention is given to environmental water allocation in light of its public good nature. Section 5 illustrates the analysis in the context of Israel's water economy and Section 6 concludes.

2 The water economy

Water can be obtained from natural or produced sources. Natural sources, indicated by the subscript n, include aquifers, lakes, reservoirs and stream flows. Produced sources include recycled (indicated by the subscript r) and desalinated water (indicated by the subscript d). Recycled water is derived from wastewater plants that collect and treat domestic and industrial sewage, where the latter is indicated by the subscript s. The user sectors are domestic (households, offices, schools, shops, hospitals etc.), agriculture (irrigation, aquaculture, livestock), industry and environment (ecosystem and ecological support). The sectors are indicated by the subscripts D (domestic), I (industry), A (agriculture) and E (environment).

Each source or sector may include multiple sub-sources or sub-sectors. For example, natural sources may be partitioned into fresh or saline,[2] surface or ground, as well as based on geographic location. Likewise, agricultural users may be divided into 2 sub-sectors, one containing growers that cannot use recycled water (e.g., growers of crops for direct human consumption) and one containing growers that can use water from all sources. To simplify the exposition, we present the analysis in the context of a water economy containing the 3 sources and 4 sectors mentioned above. The analysis is modular and can easily be extended to include sub-sources and sub-sectors.

We denote by $q_{ij}(t)$ the annual supply from source i to sector j at year t. The annual water supply from source i is thus

$$q_{i\circ}(t) = \sum_{j=A,D,I,E} q_{ij}(t),\ i = n, d, r, \tag{2.1a}$$

and the annual allocation to sector j is

$$q_{\circ j}(t) = \sum_{i=n,r,d} q_{ij}(t),\ j = D, A, I, E. \tag{2.1b}$$

2.1 Water sources

Distributing source i's water to the various sectors entails certain activities (pumping, treating, conveying) and requires capital infrastructure (pumps, pipelines, filters, sewage treatment facility, desalination plants). We discuss each source in turn.

Natural: Natural water is derived from a finite, replenishable stock $Q(t) \in [\underline{Q}, \bar{Q}]$, which evolves over time according to

$$\dot{Q}(t) = R(Q(t)) - q_{n\circ}(t), \tag{2.2}$$

[2] In Israel, natural sources with saline water (chloride concentration above 400 ml/l) is considered a separate source (see Weinberger et al. 2012).

where $R(\cdot)$ is a decreasing and concave recharge function and the upper bound $\bar{Q}$ satisfies $R(\bar{Q}) = 0$. It is straightforward to allow for multiple natural sources (aquifers, lakes, reservoirs, stream flows). In the interest of simplicity we let the stock $Q(t)$ represent the aggregate natural water stock and $R(Q)$ is the aggregate recharge.[3] The minimal water stock $\underline{Q}$, satisfying

$$Q(t) \geq \underline{Q}, \tag{2.3}$$

may represent an empty stock, in which case $\underline{Q} = 0$, or a threshold stock level below which undesirable events may occur (e.g., seawater intrusion, analyzed in Tsur and Zemel 1995). In either case (2.3) implies that when $Q(t) = \underline{Q}$ the supply of natural water cannot exceed $R(\underline{Q})$.

Preparing the natural water supply $q_{n\circ}(t)$ for distribution to end users entails certain activities (pumping, filtering, some conveyance) and requires capital infrastructure (pipelines, pumps, filters). The capital stock available for that purpose at time t is denoted $K_n(t)$.

Recycling: Health and environmental regulations require (in most places) sewage to be collected, treated and disposed without harming the environment. Consequently, a share β of the water allocated to domestic and industrial users at time t is collected in the form of sewage.[4] Thus, letting $q_{s\circ}(t)$ represent the sewage flow at time t,

$$q_{s\circ}(t) = \beta(q_{\circ D}(t) + q_{\circ I}(t)). \tag{2.4}$$

Sewage collection and treatment require capital infrastructure (treatment plants, pumps, pipelines, filters), denoted $K_s(t)$.

The treated sewage is available for reuse subject to health regulations that specify feasible

[3]If irrigation and environmental water contribute to the recharge of underlying aquifers, the recharge function takes the form $R(Q, q_{\circ A}, q_{\circ E})$, where R decreases in Q and increases in both $q_{\circ A}$ and $q_{\circ E}$.

[4]In Israel, $\beta \approx 0.65$ (see Tsur 2015).

uses for each treatment level (secondary, tertiary with different filtering methods).[5] The part of $q_{s\circ}(t)$ that is reused constitutes the supply of recycled water $q_{r\circ}(t)$. Thus,

$$q_{r\circ}(t) \leq q_{s\circ}(t). \tag{2.5}$$

Reusing the treated water requires conveyance from the recycling plants to potential users and may require further treatment (e.g., from secondary to tertiary). The capital available at time t for that purpose (conveying the recycled water from treatment facilities to potential users and possibly additional treatment required by these users) is denoted $K_r(t)$.

The distinction between $q_{s\circ}$ and $q_{r\circ}$ is needed because sewage collection and treatment, on one hand, and the allocation of the treated water to potential users, on other hand, are two separate activities. The former is (often) required by health and environmental regulation, disregarding whether the treated water is reused later on. Reusing the treated water is a policy decision that depends on the cost of conveying the recycled water from treatment plants to potential users and on the demand for the recycled water. Thus, $K_s(t)$ includes only capital needed for sewage treatment required by health and environmental regulation (see footnote 5), whereas capital needed to convey the treated water to potential users, as well as treatment beyond the level required by health and environmental regulation (if demanded by users), is included in $K_r(t)$.

The treatment level (secondary, tertiary) entails restrictions on potential uses of the recycled water. For example, secondary-treated water may not be allowed to irrigate certain crops and health regulations may prohibit the allocation of any recycled water to households, i.e.,

$$q_{rD}(t) = 0. \tag{2.6}$$

[5] Rules for recycled water use vary from country to country. In Israel, arguably the world leader in this respect, the rules distinguish between two types of recycled water: low quality (secondary treated) whose use is limited to industrial (e.g., cotton) and tree crops; and good quality (tertiary treated) that can be used in most crops. These rules are still evolving as new information is accumulated (see discussion of crop irrigation rules in Chen and Tarchitzky 2018). The current health regulation rules can be found (in Hebrew) in http://www.sviva.gov.il/subjectsEnv/Streams/SewageStandards/Pages/Milestones.aspx.

Allowing for different grades of recycled water (secondary, tertiary with different filtering technologies) requires specifying multiple recycled sources, each with its own capital stock.

Desalination: The supply of desalinated water at time t, $q_{d\circ}(t)$, is restricted by the capacity of existing desalination plants, i.e., by the available desalination capital, denoted $K_d(t)$. We assume that the quality of desalinated water permits its use by all sectors.[6]

2.2 Water sectors (users)

Distributing water to end users entails certain activities (conveying, filtering) and requires capital infrastructure dedicated for that purpose. Because water from different sources may require different conveyance systems (e.g., when mixing recycled and potable water is not allowed, as is the case in Israel), the water capital (infrastructure) used in distributing water to the various sectors is source and sector specific. Accordingly, let $K_{ij}(t)$ denote the capital infrastructure available at time t to distribute water from source $i = n, r, d$, to sector $j = D, I, A, E$.

Under (2.6), domestic users receive water only from natural (after appropriate treatment) and desalination sources. Industrial users may require water of different quality for different purposes, e.g., for cooling low quality water may suffice while beverage production requires drinking-quality water. Accounting for such restrictions entails defining multiple industrial sub-sectors, each with a separate water quality requirement.

Different agricultural users may be allowed to use water of different quality, e.g., some edible crops should not be irrigated with secondary-level recycled water (see Chen and Tarchitzky 2018), and such restrictions can be incorporated by defining multiple agricultural sub-sectors separated by their water quality restrictions.[7]

[6] This is indeed the situation in Israel, where currently desalinated water amounts to about a half of natural water supply and 65 percent of domestic water consumption.

[7] In the presence of agricultural sub-sectors, based, e.g., on the quality of irrigated water, each sub-sector has its own capital stock.

Environmental water can appear in the form of *conveyed* or *instream*. The former entails conveying water from source $i = n, r, d$, to support ecosystems and environmental sites and its allocations are the $q_{iE}(t)$'s defined above. These allocations require activities (conveyance, pumping, filtering) and the dedicated capital $K_{iE}(t)$, $i = n, r, d$. Instream water allocation amounts to avoiding (or reducing) water diversions and/or extraction from natural sources, hence entails no cost. The value of instream water, it will be shown below, is embedded in the *in situ* value of the the natural water stock Q.

2.3 Supply costs

Water supply entails variable and fixed costs. The former is associated with the cost of variable inputs (i.e., inputs that vary with the supply flow), such as temporary labor, energy and materials; the latter consists mainly of the capital cost but includes also costs of inputs that cannot be easily changed, such as tenured labor and management overheads. Both of these components vary spatially and temporally (see examples in Renzetti 1999, Harou et al. 2009, Allen et al. 2014).

Capital cost: The annual supply from source i, $q_{i\circ}(t)$, is restricted by source i's capital stock $K_i(t)$ according to

$$q_{i\circ}(t) \leq \gamma_i K_i(t),\ i = n, r, d, s, \tag{2.7a}$$

where the γ_i's are capital utilization parameters, indicating the maximal annual supply per unit capital K_i.[8] Similarly, the allocation of water from source i to sector j at time t, $q_{ij}(t)$, is restricted by the capital infrastructure $K_{ij}(t)$ according to

$$q_{ij}(t) \leq \gamma_{ij} K_{ij}(t),\ i = n, r, d,\ j = D, I, A, E, \tag{2.7b}$$

[8]Including the sewage flow $q_{s\circ}(t)$ in the list of sources simplifies notation. Notice that $q_{s\circ}(t)$, which is the outcome of domestic and industrial allocations (cf. (2.4)), is the source of recycled water (cf. (2.5)).

where the γ_{ij}'s are capital utilization parameters, indicating the maximal annual supply from source i to sector j per unit K_{ij}.

The capital stocks $K_i(t)$ and $K_{ij}(t)$ evolve in time according to

$$\dot{K}_i(t) = x_i(t) - \delta K_i(t),\ i = n, r, d, s, \tag{2.8a}$$

$$\dot{K}_{ij}(t) = x_{ij}(t) - \delta K_{ij}(t),\ i = n, r, d,\ j = D, I, A, E, \tag{2.8b}$$

where $x_i(t)$ and x_{ij} represent investment rates and δ is a constant depreciation rate.[9] Aggregate investment at time t, denoted $X(t)$, is bounded by an exogenous investment budget $\bar{x}$:

$$X(t) \equiv \sum_{i=n,r,d,s} x_i(t) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} x_{ij}(t) \leq \bar{x} \tag{2.9}$$

for all $t \geq 0$.

The annual cost of capital consists of the total investment $X(t)$. We shall see below that this cost equals the finance (i.e., interest payments) and depreciation costs associated with the aggregate capital stock.

Variable costs: The variable cost associated with $q_{i\circ}$, $i = n, r, d, s$, are represented by the increasing and convex functions $C_i(q_{i\circ})$, $i = n, r, d, s$. For natural water ($i = n$), $C_n(\cdot)$ may depend also on the stock of natural water Q, in which case $C_n(Q, q_{n\circ})$ is non-increasing and concave in Q, e.g., $C_n(Q, q_{n\circ}) = c_n(Q)q_{n\circ}$, where the unit extraction cost function $c_n(Q)$ is non-increasing and convex. Likewise the variable costs associated with distributing source i's water to sector j are represented by $C_{ij}(q_{ij}(t))$, $i = n, r, d,\ j = D, I, A, E$. The variable cost associated with the allocation $q(t) = \{q_{i,j}(t), i = n, r, d,\ j = D, I, A, E\}$ is therefore

$$C(Q(t), q(t) = C_n(Q(t), q_{n\circ}(t)) + \sum_{i=r,d,s} C_i(q_{i\circ}(t)) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} C_{ij}(q_{ij}(t)), \tag{2.10}$$

where it is recalled that the sewage flow $q_{s\circ}(t)$ is defined in (2.4).

[9]For simplicity, all capital stocks are assumed to depreciate at the same rate δ.

Let

$$c_i(q_{i\circ}) = \partial C_i(q_{i\circ})/\partial q_{i\circ},\ i = n, r, d, s,\ c_{ij}(q_{ij}) = \partial C_j(q_{ij})/\partial q_{ij},\ i = n, r, d,\ j = D, I, A, E, \tag{2.11}$$

where $c_n(Q, q_{n\circ})$ and $c_n(q_{n\circ})$ are used interchangeably. Differentiating $C(Q(t), q(t))$ with respect to $q_{ij}(t)$, noting (2.4), gives the following marginal costs of water allocation from source i to sector j:

$$c_i(q_{i\circ}(t)) + c_{ij}(q_{ij}(t)) + \beta c_s(q_{s\circ}(t)),\ i = n, r, d,\ j = D, I, \tag{2.12a}$$

$$c_i(q_{i\circ}(t)) + c_{ij}(q_{ij}(t)),\ i = n, r, d,\ j = A, E. \tag{2.12b}$$

2.4 Water demand

Sector j's annual (inverse) demand for water is denoted $D_j(q_{\circ j})$, $j = D, A, I, E$.[10] This curve measures the annual quantity of water demanded by sector j at any water price and can be interpreted as the price sector j's users are willing to pay for an additional (marginal) unit of water when they already consume $q_{\circ j}$. This interpretation allows defining the demand for conveyed environmental water $D_E(q_{\circ E})$, similar to the other sectors, as the willingness to pay (WTP) for an additional water unit at any level of (conveyed) environmental water supply. However, while the other sectors' demands can be estimated by price-quantity data, the public good nature of environmental water eliminates this possibility and estimating the demand $D_E(\cdot)$ requires indirect, WTP elicitation methods.[11]

[10] It is assumed that the minimal annual water flow required for basic human needs is provided at a nominal or no cost. The domestic sector's demand refers to annual flows above this subsistent level.

[11] There is a large literature on sectoral water demands. Examples of agricultural water demand include Just et al. (1983), Moore et al. (1994), Howitt (1995), Mundlak (2001), Tsur et al. (2004), Schoengold et al. (2006), Scheierling et al. (2006); examples of urban and industrial demands include Baumann et al. (1997), Renzetti (2002, 2015), Worthington and Hoffmann (2006), Olmstead et al. (2007), House-Peters and Chang (2011), Baerenklau et al. (2014), Smith and Zhao (2015); examples of environmental water demand include Dudley and Scott (1997), Loomis et al. (2000), Pimentel et al. (2004), Fleischer and Tsur (2009), Thiene and Tsur (2013), Koundouri and Davila (2015), Koundouri et al. (2017) and references they cite.

The annual gross surplus of sector j generated by $q_{\circ j}$ (before subtracting the cost of water supply) is the area underneath the demand curve to the left of $q_{\circ j}$:

$$B_j(q_{\circ j}) = \int_0^{q_{\circ j}} D_j(s)ds, \; j = D, I, A, E. \tag{2.13}$$

The surplus $B_D(q_{\circ D})$ represents households's benefit associated with the direct use of water (drinking, cooking, hygiene, gardening). In addition, households enjoy the environmental benefit $B_E(q_{\circ E})$ generated by the conveyed environmental water allocation and the benefit associated with the instream water use provided by the natural water stock Q. The latter (instream) benefit is represented by the nondecreasing and concave function $B_E^{is}(Q)$. The annual benefit of all sectors (before subtracting the supply cost) is therefore

$$B(Q(t), q(t)) = \sum_{j=D,I,A,E} B_j(q_{\circ j}(t)) + B_E^{is}(Q(t)). \tag{2.14}$$

Subtracting the variable costs of supply and the investment expenditures gives the net (annual) benefit flow:

$$B(Q(t), q(t)) - C(Q(t), q(t)) - X(t)$$

where $B(Q(t), q(t))$, $C(Q(t), q(t))$ and $X(t)$ are defined in (2.14), (2.10) and (2.9), respectively, and $q(t) = \{q_{ij}(t), i = n, r, d, j = D, I, A, E\}$ and $x(t) = \{x_i, i = n, r, d, s; x_{ij}(t), i = n, r, d, j = D, I, A, E\}$ represent the water allocations and investments at time t.

2.5 Water policy

A water policy consists of the water allocations $q(t)$ and investment rates $x(t)$ throughout the (indefinite) planning horizon $t \geq 0$. Given the initial natural water stock $Q(0)$ and capital stocks $K(0) = \{K_i(0), i = n, r, d, s, \; K_{ij}(t), i = n, r, d, j = D, I, A, E\}$, a water policy determines the evolution of $Q(t)$ and $K(t)$ via (2.2) and (2.8), respectively, and generates the payoff

$$\int_0^{\infty} (B(Q(t), q(t)) - C(Q(t), q(t)) - X(t))\, e^{-\rho t} dt, \tag{2.15}$$

where ρ is the time rate of discount, reflecting time preferences as well as the cost of capital.

Noting (2.8) and (2.9), the term involving $X(t)$ in (2.15) can be expressed as

$$\int_0^\infty X(t)e^{-\rho t}dt = \int_0^\infty \left(\sum_{i=n,r,d,s} \left(\dot{K}_i(t) + \delta K_i(t)\right) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} \left(\dot{K}_{ij}(t) + \delta, K_{ij}(t)\right) \right) e^{-\rho t}dt.$$

Integrating by parts the right-hand side gives

$$\int_0^\infty X(t)e^{-\rho t}dt = \int_0^\infty (\rho + \delta)\mathbf{K}(t)e^{-\rho t}dt + \mathbf{K}(0), \tag{2.16}$$

where

$$\mathbf{K}(t) = \sum_{i=n,r,d,s} K_i(t) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} K_{ij}(t) \tag{2.17}$$

is the total water capital (infrastructure) at time t and $\mathbf{K}(0)$ is its initial level. The term $(\rho + \delta)\mathbf{K}$ represents the annual cost on a capital stock worth $\mathbf{K}$, consisting of the interest ($\rho\mathbf{K}$) and depreciation ($\delta\mathbf{K}$) costs. In view of (2.16), the payoff (2.15) can be expressed, ignoring the (exogenously given) initial total capital $\mathbf{K}(0)$, as

$$\int_0^\infty \left[B(Q(t), q(t)) - C(Q(t), q(t)) - (\rho + \delta)\mathbf{K}(t)\right] e^{-\rho t}dt. \tag{2.18}$$

3 Optimal policy

The optimal policy is the feasible $\{q(t), x(t),\, t \geq 0\}$ that maximize (2.18) subject to the state dynamics (2.2) and (2.8), given the initial natural water stock $Q(0)$ and capital stocks $K(0)$, where feasibility entails conditions (2.3), (2.5), (2.6), (2.7), (2.9) and nonnegativity of $q(t)$ and $x(t)$. The multiple states associated with the elements of $K(t)$ complicate analytical characterization of the optimal policy. Tsur and Zemel (2018) simplify this characterization by showing that it can be broken into two subproblems, each containing the natural water stock $Q(t)$ as a single state: the first subproblem ignores the investment budget constraint (2.9) and the second accounts for it. The optimal policy corresponding to the first and second subproblems are called *turnpike* and *most-rapid-approach* (MRAP), respectively. The optimal policy begins with the MRAP policy, during which the water infrastructure is constructed and the investment budget $\bar{x}$ is fully utilized (i.e., constraint (2.9) holds as equality) until the turnpike policy becomes feasible, at which time the optimal policy switches to the turnpike policy and evolves along it thereafter, eventually converging to a steady state. I summarize below the optimal policy, starting with the turnpike.

3.1 The turnpike policy

Suppose that the infrastructure configuration $K(t) = \{K_i(t), i = n, r, d, s,\ K_{ij}(t), i = n, r, d, j = D, I, A, E\}$ can be freely chosen. The problem then is to choose the feasible $q(t)$ and $K(t)$ that maximize (2.18) subject to (2.2) given $Q(0)$, where feasibility entails (2.3), (2.5), (2.6), (2.7) and nonnegativity of $q(t)$ and $K(t)$. The corresponding optimal policy and processes are called *turnpike* and denoted with a "tilde" overhead, e.g., $\tilde{q}(t)$, $\tilde{K}(t)$ and $\tilde{Q}(t)$. We note that the turnpike processes depend on the initial natural water stock and the latter is added as an argument when needed, e.g., $\tilde{K}(t; Q)$ represents the turnpike capital process that departs from the initial natural water stock $Q(0) = Q$.

Observing (2.18), it is seen that treating the capital stocks as decision variables, the $\tilde{K}(t) = \{\tilde{K}_i(t), \tilde{K}_{ij}(t)\}$ are the capital stocks that would have been chosen if it were possible to rent capital at the (competitive) rental rate $\rho + \delta$. In view of (2.7), supplying one water unit from source i to sector j requires at least $1/\gamma_i$ units of K_i and $1/\gamma_{ij}$ units of K_{ij}. In addition, each cubic meter allocated to domestic or industrial users ($j = D, I$) requires also the sewage capital $\beta(1/\gamma_s)$ needed to collect and treat the ensuing domestic and industrial sewage (cf. (2.4)). Under the capital rental rate $\rho + \delta$, the capital costs per unit water supplied from source $i = n, r, d$, to sector j are therefore

$$\mu_i + \mu_{ij} + \beta\mu_s,\ j = D, I, \tag{3.1a}$$

$$\mu_i + \mu_{ij},\ j = A, E, \tag{3.1b}$$

where

$$\mu_i \equiv (\rho+\delta)/\gamma_i,\ i=n,r,d,s, \tag{3.2a}$$

$$\mu_{ij} \equiv (\rho+\delta)/\gamma_{ij},\ i=n,r,d,\ j=D,I,A,E, \tag{3.2b}$$

are the unit capital costs (i.e., annual cost of capital per unit water).

The marginal costs are specified in (2.12). Additional costs are associated with the shadow (scarcity) price of the natural water stock, denoted $\tilde{\theta}(t)$, and the shadow price of recycled water, associated with constraint (2.5), denoted $\tilde{\xi}(t)$. The former applies only to water derived from natural sources (i.e., q_{nj}); the latter applies as a price (tax) for consumers of recycled water and as a subsidy for contributors to recycled water (domestic and industrial users). We now list the necessary conditions characterizing the turnpike policy (see appendix for derivation).

Natural water allocated to domestic or industrial users ($\tilde{q}_{nj},\ j=D,I$):

$$D_j(\tilde{q}_{\circ j}(t)) \le c_n(\tilde{Q}(t),\tilde{q}_{n\circ}(t))+c_{nj}(\tilde{q}_{nj}(t))+\mu_n+\mu_{nj}+\beta[c_s(\tilde{q}_{s\circ}(t))+\mu_s]+\tilde{\theta}(t)-\beta\tilde{\xi}(t), \tag{3.3a}$$

equality holding if $\tilde{q}_{nj}(t)>0,\ j=D,I$. Natural water allocated to agricultural users or the environment ($\tilde{q}_{nj},\ j=A,E$):

$$D_j(\tilde{q}_{\circ j}(t)) \le c_n(\tilde{Q}(t),\tilde{q}_{n\circ}(t))+c_{nj}(\tilde{q}_{nj}(t))+\mu_n+\mu_{nj}+\tilde{\theta}(t), \tag{3.3b}$$

equality holding if $\tilde{q}_{nj}(t)>0,\ j=A,E$. Recycled water allocated to industrial users ($\tilde{q}_{rI}$):[12]

$$D_I(\tilde{q}_{\circ I}(t)) \le c_r(\tilde{q}_{r\circ}(t))+c_{rI}(\tilde{q}_{rI}(t))+\mu_r+\mu_{rI}+\beta[c_s(\tilde{q}_{s\circ}(t))+\mu_s]+\tilde{\xi}(t)(1-\beta), \tag{3.4a}$$

equality holding if $\tilde{q}_{rI}(t)>0$. Recycled water allocated to agricultural users or the environment ($\tilde{q}_{rj},\ j=A,E$):

$$D_j(\tilde{q}_{\circ j}(t)) \le c_r(\tilde{q}_{r\circ}(t))+c_{rj}(\tilde{q}_{rj}(t))+\mu_r+\mu_{rj}+\tilde{\xi}(t), \tag{3.4b}$$

[12] Assuming (2.6), recycled water to households is not allowed, while industrial users can use recycled water. If some industrial users can use only drinking quality water, an additional sector consisting of this industrial sub-sector should be defined.

equality holding if $\tilde{q}_{rj}(t) > 0$, $j = A, E$. Desalinated water allocated to domestic or industrial users ($\tilde{q}_{dj}$, $j = D, I$):

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_d(\tilde{q}_{d\circ}(t)) + c_{dj}(\tilde{q}_{dj}(t)) + \mu_d + \mu_{dj} + \beta[c_s(\tilde{q}_{s\circ}(t)) + \mu_s] - \beta\tilde{\xi}(t), \tag{3.5a}$$

equality holding if $\tilde{q}_{d\,j}(t) > 0$, $j = D, I$. Desalinated water allocated to agricultural users or the environment ($\tilde{q}_{dj}$, $j = A, E$):

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_d(\tilde{q}_{d\circ}(t)) + c_{dj}(\tilde{q}_{dj}(t)) + \mu_d + \mu_{dj}, \tag{3.5b}$$

equality holding if $\tilde{q}_{d\,j}(t) > 0$, $j = A, E$.

The shadow price of natural water, $\tilde{\theta}(t)$, evolves in time according to

$$\dot{\tilde{\theta}}(t)) - \rho\tilde{\theta}(t) = C_{nQ}(\tilde{Q}(t), \tilde{q}_{n\circ}(t)) - B_E^{is\prime}(\tilde{Q}(t)) - \tilde{\theta}(t)R'(\tilde{Q}(t)) - \tilde{\vartheta}(t), \tag{3.6}$$

where $C_{nQ} \equiv \partial C_n/\partial Q$,[13] $R' \equiv \partial R/\partial Q$, $B_E^{is\prime} \equiv \partial B_E^{is}/\partial Q$ and $\tilde{\vartheta}(t)$ is the scarcity price of natural water, i.e., the shadow price of constraint (2.3). The multipliers $\tilde{\xi}(t)$ and $\tilde{\vartheta}(t)$ satisfy the complementary slackness conditions

$$\tilde{\xi}(t)[\tilde{q}_{s\circ}(t) - \tilde{q}_{r\circ}(t)] = 0, \tag{3.7a}$$

and

$$\tilde{\vartheta}(t)[\tilde{Q}(t) - \underline{Q}] = 0. \tag{3.7b}$$

Finally, no idle capital is allowed along the turnpike, i.e., the capital constraints (2.7) are binding, implying

$$\tilde{K}_i(t) = \tilde{q}_{i\circ}(t)/\gamma_i,\ i = n, r, d, s, \tag{3.8a}$$

$$\tilde{K}_{ij}(t) = \tilde{q}_{ij}(t)/\gamma_{ij},\ i = n, r, d,\ j = D, I, A, E. \tag{3.8b}$$

Conditions (3.3)-(3.5) are in essence demand-equals-supply conditions, with demand on the left-hand sides and unit supply cost on the right-hand sides. Let us denote the the unit

[13]E.g., if $C_n(Q, q_{n\circ}) = c_n(Q)q_{n\circ}$, with $c_n(\cdot)$ a non-increasing unit cost function, then $C_{nQ} = c_n'(Q)q_{n\circ}$.

supply costs along the turnpike, i.e., right-hand sides of (3.3)-(3.5), by $\tilde{p}_{ij}(t)$. We repeat these unit supply costs and identify their components:

$$\tilde{p}_{nj}(t) = \underbrace{c_n(\tilde{Q}(t), \tilde{q}_{n\circ}(t))}_{\text{mar cost } n} + \underbrace{c_{nj}(\tilde{q}_{nj}(t))}_{\text{mar cost } nj} + \underbrace{\mu_n}_{\text{cap cost } K_n} + \underbrace{\mu_{nj}}_{\text{cap cost } K_{nj}} + \underbrace{\beta c_s(\tilde{q}_{s\circ}(t))}_{\text{mar cost } s} + \underbrace{\beta\mu_s}_{\text{cap cost } K_s} + \underbrace{\tilde{\theta}(t)}_{\text{shadow price } Q} - \underbrace{\beta\tilde{\xi}(t)}_{\text{rec subsidy}}, \ j = D, I; \quad (3.9a)$$

$$\tilde{p}_{nj}(t) = \underbrace{c_n(\tilde{Q}(t), \tilde{q}_{n\circ}(t)))}_{\text{mar cost } n} + \underbrace{c_{nj}(\tilde{q}_{nj}(t))}_{\text{mar cost } nj} + \underbrace{\mu_n}_{\text{cap cost } K_n} + \underbrace{\mu_{nj}}_{\text{cap cost } K_{nj}} + \underbrace{\tilde{\theta}(t)}_{\text{shadow price } Q}, \ j = A, E; \quad (3.9b)$$

$$\tilde{p}_{rI}(t) = \underbrace{c_r(\tilde{q}_{r\circ}(t))}_{\text{mar cost } r} + \underbrace{c_{rI}(\tilde{q}_{rI}(t))}_{\text{mar cost } rI} + \underbrace{\mu_r}_{\text{cap cost } K_r} + \underbrace{\mu_{rI}}_{\text{cap cost } K_{rI}} + \underbrace{\beta c_s(\tilde{q}_{s\circ}(t))}_{\text{mar cost s}} + \underbrace{\beta\mu_s}_{\text{cap cost } K_s} + \underbrace{\xi(t)(1-\beta)}_{\text{rec tax-subsidy}}; \quad (3.10a)$$

$$\tilde{p}_{rj}(t) = \underbrace{c_r(\tilde{q}_{r\circ}(t))}_{\text{mar cost } r} + \underbrace{c_{rj}(\tilde{q}_{rj}(t))}_{\text{mar cost } rj} + \underbrace{\mu_r}_{\text{cap cost } K_r} + \underbrace{\mu_{rj}}_{\text{cap cost } K_{rj}} + \underbrace{\tilde{\xi}(t)}_{\text{rec tax}}, \ j = A, E; \quad (3.10b)$$

$$\tilde{p}_{dj}(t) = \underbrace{c_d(\tilde{q}_{d\circ}(t))}_{\text{mar cost } d} + \underbrace{c_{dj}(\tilde{q}_{dj}(t))}_{\text{mar cost } dj} + \underbrace{\mu_d}_{\text{cap cost } K_d} + \underbrace{\mu_{dj}}_{\text{cap cost } K_{dj}} + \underbrace{\beta c_s(\tilde{q}_{s\circ}(t))}_{\text{mar cost } s} + \underbrace{\beta\mu_s}_{\text{cap cost } K_s} - \underbrace{\tilde{\xi}(t)\beta}_{\text{rec subsidy}}, \ j = D, I; \quad (3.11a)$$

$$\tilde{p}_{dj}(t) = \underbrace{c_d(\tilde{q}_{d\circ}(t))}_{\text{mar cost } d} + \underbrace{c_{dj}(\tilde{q}_{dj}(t))}_{\text{mar cost } dj} + \underbrace{\mu_d}_{\text{cap cost } K_d} + \underbrace{\mu_{dj}}_{\text{cap cost } K_{dj}}, \ j = A, E. \quad (3.11b)$$

In terms of the above $\tilde{p}_{ij}(t)$, Conditions (3.3)-(3.5) can be succinctly rendered as

$$D_j(\tilde{q}_{\circ j}(t)) \le \tilde{p}_{ij}(t) \text{ equality holding if } \tilde{q}_{ij}(t) > 0, \ i = n, r, d, \ j = D, I, A, E. \quad (3.12)$$

A steady-state detour

The turnpike processes change over time as long as the natural water stock $\tilde{Q}(t)$ changes. If $\tilde{Q}(t)$ remains constant, all turnpike processes stay put as well. This happens to be the case if $\tilde{Q}(t)$ converges to a steady state. Because the turnpike problem is a single-state, infinite horizon and autonomous, the turnpike processes eventually converge to a steady (see Tsur and Zemel 2014). We denote steady state values with a hat " ˆ " overhead notation. Condition (2.2) implies

$$R(\hat{Q}) = \hat{q}_{n\circ} \tag{3.13}$$

and condition (3.6) specializes to[14]

$$\hat{\theta} = \frac{\hat{\vartheta} - c_n'(\hat{Q})R(\hat{Q}) + B_E^{is\prime}(\hat{Q})}{\rho - R'(\hat{Q})}. \tag{3.14}$$

Conditions (3.12) specialize in a steady state to

$$D_j(\hat{q}_{\circ j}) \leq \hat{p}_{ij} \text{ equality holding if } \hat{q}_{ij} > 0,\ i = n, r, d,\ j = D, I, A, E, \tag{3.15}$$

where $\hat{q}_{rD} = 0$ if (2.6) is imposed, and (3.7) become

$$\hat{\xi}[\hat{q}_{s\circ} - \hat{q}_{r\circ}] = 0, \tag{3.16a}$$

$$\hat{\vartheta}(\hat{Q} - \underline{Q}) = 0, \tag{3.16b}$$

where $\hat{q}_{s\circ}$ follows form (2.4).

Equation (3.14) illuminates the three components comprising the shadow price of natural water: scarcity, extraction cost and instream value. The scarcity cost component is represented by $\hat{\vartheta}$ – the shadow price of the $Q(t) \geq \underline{Q}$ constraint. This term is positive when the steady state falls on the lower bound $\underline{Q}$, in which case extraction, $\hat{q}_{n\circ}$, cannot exceed the recharge $R(\hat{Q}) = R(\underline{Q})$ and the cost of this constraint is embedded in $\hat{\vartheta}$. The extraction cost component is represented by the $-c_n'(\hat{Q})R(\hat{Q})$ term, which is nonnegative because $c(Q)$ is non-increasing. Finally, the instream value of Q affects its shadow price via the marginal instream benefit term $B_E^{is\prime}(\hat{Q})$. When positive, this term increases $\hat{\theta}$, rendering natural water more expensive, thereby reducing diversion and/or pumping from natural sources.

[14] We assume that $C_n(Q, q_{n\circ}) = c_n(Q)q_{n\circ}$, so $C_{nQ}(Q, q_{n\circ}) \equiv \partial C_n(Q, q_{n\circ})/\partial Q = c_n'(Q)q_{n\circ}$.

Equations (3.13)-(3.16) provide 16 conditions (including (2.6) if $\hat{q}_{rD}(t) = 0$ is imposed) to solve for $\hat{q} = \{\hat{q}_{ij},\, i = n, r, d,\, j = D, I, A, E\}$, $\hat{Q}$, $\hat{\theta}$, $\hat{\xi}$ and $\hat{\vartheta}$ as follows:

Property 1 (Steady state). (*i*) *if equations (3.15)-(3.16a) admit nonnegative solutions for* $\hat{q}$, $\hat{Q}$, $\hat{\theta}$ *and* $\hat{\xi}$ *with* $\hat{Q} > \underline{Q}$ *and* $\hat{\vartheta} = 0$, *then these are the steady state values.* (*ii*) *If no such solutions exist, then* $\hat{Q} = \underline{Q}$ *and* $\hat{q}$, $\hat{\xi}$ *and* $\hat{\vartheta}$ *are the nonnegative solutions of (3.15)-(3.16).* (*iii*) *Noting (3.8), the steady state capital stocks are*

$$\hat{K}_i = \hat{q}_{i\circ}/\gamma_i,\; i = n, r, d, s, \tag{3.17a}$$

$$\hat{K}_{ij} = \hat{q}_{ij}/\gamma_{ij},\; i = n, r, d,\; j = D, I, A, E. \tag{3.17b}$$

Back to the turnpike

The turnpike policy can now be characterized as follows:

Property 2 (Turnpike). *Given* $\tilde{Q}(t)$ *and* $\tilde{\theta}(t)$, *the turnpike allocations* $\tilde{q}(t)$ *and the shadow price of recycled water* $\tilde{\xi}(t)$ *solve (3.12) and (3.7a). The state and costate processes,* $\tilde{Q}(t)$ *and* $\tilde{\theta}(t)$, *are solved from (2.2), (3.6) and (3.7b), given the initial* $\tilde{Q}(0) = Q(0)$ *and boundary (long run) values* $\hat{Q}$ *and* $\hat{\theta}$, *specified in Property 1. The ensuing capital processes,* $\tilde{K}(t)$, *are specified in (3.8).*

Under mild smoothness conditions (see Tsur and Zemel 2018), the $\tilde{K}_i(t)$ and $\tilde{K}_{ij}(t)$ are differentiable in time and the corresponding turnpike investment processes

$$\tilde{x}_i(t) = \dot{\tilde{K}}_i(t) + \delta\tilde{K}_i(t),\; i = n, r, d, s, \tag{3.18a}$$

$$\tilde{x}_{ij}(t) = \dot{\tilde{K}}_{ij}(t) + \delta\tilde{K}_{ij}(t),\; i = n, r, d,\; j = D, I, A, E, \tag{3.18b}$$

induced by (2.8), are well defined. The turnpike capital stocks $\tilde{K}_i(t)$ and $\tilde{K}_{ij}(t)\}$ can be viewed as driven (constructed) by the investments $\tilde{x}_i(t)$ and $\tilde{x}_{ij}(t)$. It is assumed that the investment budget $\bar{x}$ can support the turnpike investments at all times, i.e.,

$$\tilde{X}(t) \equiv \sum_{i=n,r,d,s} \tilde{x}_i(t) + \sum_{i=n,r,d}\; \sum_{j=D,I,A,E} \tilde{x}_{ij}(t) < \bar{x},\; t \geq 0. \tag{3.19}$$

3.2 The MRAP (infrastructure construction) policy

The turnpike policy departs from the initial capital stocks

$$\tilde{K}(0) = \{\tilde{K}_i(0), i = n, r, d, s, \tilde{K}_{ij}(0), i = n, r, d, j = D, I, A, E\},$$

defined in (3.8) in terms of the initial turnpike water allocations $\tilde{q}(0) = \{\tilde{q}_{ij}(0), i = n, r, d, j = D, I, A, E\}$. In actual practice, the (actual) initial capital $K(0)$ is what it is and may differ from $\tilde{K}(0)$. If the total initial capital $\mathbf{K}(0) = \sum_{i=n,r,d,s} K_i(0) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} K_{ij}(0)$ falls below $\tilde{\mathbf{K}}(0) = \sum_{i=n,r,d,s} \tilde{K}_i(0) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} \tilde{K}_{ij}(0)$, then initially the turnpike policy is not feasible. In this cases, it is optimal to construct the capital infrastructure as rapidly as possible, by utilizing the entire investment budget, until the turnpike policy becomes feasible, at which time the optimal policy switches to the turnpike policy and evolves along it thereafter. If the investment budget $\bar{x}$ is unlimited, it is possible to increase the capital stocks to the desired, turnpike configuration instantly. If, however, $\bar{x}$ is finite, then building the capital infrastructure takes time. The infrastructure construction period (policy) is denoted the MRAP period (policy). The name stems from the property that the investment budget is fully utilized during this period, hence the approach to the turnpike is as rapidly as possible. The terms "MRAP policy" and "construction policy" are used interchangeably.

The MRAP policy differs from the turnpike policy in that the total capital stock $\mathbf{K}(t)$, defined in (2.17), is restricted not to exceed the maximal feasible total capital stock $\bar{\mathbf{K}}(t)$ obtained under the maximal total investment $\bar{x}$. Noting (2.8), the total capital $\mathbf{K}(t)$ evolves in time according to

$$\dot{\mathbf{K}}(t) = X(t) - \delta \mathbf{K}(t) \tag{3.20}$$

where $X(t)$ is the total investment (cf. (2.9)). The frontier process $\bar{\mathbf{K}}(t)$, obtained under $X(t) = \bar{x}$, satisfies

$$\dot{\bar{\mathbf{K}}}(t) = \bar{x} - \delta \bar{\mathbf{K}}(t), \tag{3.21}$$

implying that $\bar{\mathbf{K}}(t)$ evolves in time according to

$$\bar{\mathbf{K}}(t) = \frac{\bar{x}}{\delta}\left(1 - e^{-\delta t}\right) + \mathbf{K}(0)e^{-\delta t}, \tag{3.22}$$

where $\mathbf{K}(0)$ is the (actual) initial total capital. Clearly, any feasible capital process satisfies

$$\mathbf{K}(t) \le \bar{\mathbf{K}}(t) \tag{3.23}$$

The *restricted turnpike* problem is obtained by adding constraint (3.23) to the turnpike problem. The corresponding optimal policy, denoted the restricted turnpike policy, thus, entails finding the feasible $q(t)$ and $K(t)$ that maximize (2.18) subject to (2.2) given $Q(0)$, where feasibility entails (2.3), (2.5), (2.6), (2.7), nonnegativity of $q(t)$ and $K(t)$ and (3.23). The optimal processes corresponding to the restricted turnpike problem are called the *restricted turnpike* processes and denoted with a "double-tilde" overhead, e.g., $\tilde{\tilde{Q}}(t)$, $\tilde{\tilde{q}}(t)$ and $\tilde{\tilde{K}}(t)$. Because the restricted turnpike problem is, as the name suggests, a restricted version of the turnpike problem, it follows that:

Claim 1. *If the turnpike policy is feasible for the restricted turnpike problem, it must be optimal for the restricted turnpike problem.*

The MRAP policy coincides with the restricted turnpike policy when constraint (3.23) is binding, i.e., while the entire investment budget is utilized. Formulating this policy, thus, requires identifying the period during which the constraint is binding. To that end, we use the property that the turnpike problem is infinite-horizon and autonomous to express the turnpike processes as functions of the (natural water) state (see, e.g., Leonard and Long 1992, pp. 289-294), i.e., the turnpike water allocations and capital stocks at time t when the natural water stock is $Q(t)$ are represented by $\tilde{q}(Q(t))$ and $\tilde{K}(Q(t))$ and the total capital stock is

$$\tilde{\mathbf{K}}(Q(t)) = \sum_{i=n,r,d,s} \tilde{K}_i(Q(t)) + \sum_{i=n,r,d}\ \sum_{j=D,I,A,E} \tilde{K}_{ij}(Q(t)).$$

Likewise, the shadow price of the natural water stock at time t is expressed as $\tilde{\theta}(Q(t))$.

Clearly, if $\tilde{\mathbf{K}}(\tilde{\tilde{Q}}(t_0)) \leq \bar{\mathbf{K}}(t_0)$ at some time t_0, then $\tilde{\mathbf{K}}(\tilde{\tilde{Q}}(t)) \leq \bar{\mathbf{K}}(t)$ for all $t \geq t_0$. This is so because, noting (3.20)-(3.21), $\tilde{\mathbf{K}}(\tilde{\tilde{Q}}(t))$ is driven by the total turnpike investments whereas the frontier process $\bar{\mathbf{K}}(t)$ is driven by the investment budget $\bar{x}$ and the latter exceeds the former (cf. (3.19)). Thus, if $\tilde{\mathbf{K}}(Q(0)) \leq \bar{\mathbf{K}}(0) = \mathbf{K}(0)$, then (3.19) implies that constraint (3.23) is never binding and the turnpike policy is feasible from the outset, hence, according to Claim 1, is also optimal. In this case, the MRAP (construction) policy is never implemented.

If $\mathbf{K}(0) < \tilde{\mathbf{K}}(Q(0))$, the turnpike policy is initially not feasible. In this case, the MRAP policy is implemented until the turnpike policy becomes feasible, at which time the optimal policy switches to the turnpike polity. During the construction (MRAP) period, the total capital process evolves along $\bar{\mathbf{K}}(t)$, specified in (3.22). The turnpike policy becomes feasible at time τ, defined as the time at which $\bar{\mathbf{K}}(t)$ reaches the total capital associated with the turnpike policy that departs from the current natural water stock, i.e., τ satisfies

$$\bar{\mathbf{K}}(\tau) = \tilde{\mathbf{K}}(\tilde{\tilde{Q}}(\tau)). \tag{3.24}$$

We summarize the above discussion in:

Property 3. *(i) If $\mathbf{K}(0) \geq \tilde{\mathbf{K}}(Q(0))$, the turnpike policy is feasible, hence optimal, from the outset. (ii) If $\mathbf{K}(0) < \tilde{\mathbf{K}}(Q(0))$, the turnpike policy is initially not feasible. In this case, the MRAP policy, under which total investment equals the budget $\bar{x}$, is implemented until time τ, at which time the optimal policy switches to the turnpike policy that departs form the natural water stock $\tilde{\tilde{Q}}(\tau)$ and proceeds along it thereafter.*

It remains to characterize the MRAP policy, i.e., the optimal construction policy during $t \in [0, \tau)$, while constraint (3.23) is binding and the turnpike policy is infeasible. To that end, let us define

$$\begin{cases} m_i(t) = \mu_i + \eta(t)/\gamma_i, & i = n, r, d, s, \\ m_{ij}(t) = \mu_{ij} + \eta(t)/\gamma_{ij}, & i = n, r, d, j = D, I, A, E, \end{cases} \tag{3.25}$$

where the unit capital costs μ_i and μ_{ij} are specified in (3.2) and $\eta(t) \geq 0$ is the shadow price of constraint (3.23). Let $\tilde{\tilde{p}}_{ij}(t), i = n, r, d,\ j = D, I, A, E$, be the $\tilde{p}_{ij}(t)$, defined in (3.9)-(3.11), with $m_i(t)$ and $m_{ij}(t)$ substituting μ_i and μ_{ij}, respectively. Replacing $\tilde{\tilde{p}}_{ij}(t)$ for $\tilde{p}_{ij}(t)$ in (3.12) gives the following necessary conditions for the $\tilde{\tilde{q}}(t)$ processes:

$$D_j(\tilde{\tilde{q}}_{\circ j}(t)) \leq \tilde{\tilde{p}}_{ij}(t) \text{ equality holding if } \tilde{\tilde{q}}_{ij}(t) > 0,\ i = n, r, d,\ j = D, I, A, E. \tag{3.26}$$

Likewise, $\tilde{\tilde{\theta}}(t)$ replaces $\tilde{\theta}(t)$ in (3.6) and $\tilde{\tilde{\xi}}(t)$, $\tilde{\tilde{\vartheta}}(t)$ replace $\tilde{\xi}(t)$, $\tilde{\vartheta}(t)$ in (3.7). The MRAP policy can now be characterized as follows (see proof in the appendix):

Property 4. *(i) During $t \in [0, \tau]$, the $\tilde{\tilde{q}}(t)$ and $\tilde{\tilde{\xi}}(t)$ processes solve (3.26) and (3.7a) given $\tilde{\tilde{Q}}(t)$ and $\tilde{\tilde{\theta}}(t)$, with $\eta(t) \geq 0$ chosen such that*

$$\tilde{\tilde{\boldsymbol{K}}}(t) = \bar{\boldsymbol{K}}(t),\ t \in [0, \tau], \tag{3.27}$$

where

$$\tilde{\tilde{K}}_i(t) = \tilde{\tilde{q}}_{i\circ}(t)/\gamma_i,\ i = n, r, d, s, \tag{3.28a}$$

$$\tilde{\tilde{K}}_{ij}(t) = \tilde{\tilde{q}}_{ij}(t)/\gamma_{ij},\ i = n, r, d,\ j = D, I, A, E, \tag{3.28b}$$

$\tilde{\tilde{\boldsymbol{K}}}(t) = \sum_{i=n,r,d,s} \tilde{\tilde{K}}_i(t) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} \tilde{\tilde{K}}_{ij}(t)$, *and τ satisfies (3.24). (ii) The state and costate processes, $\tilde{\tilde{Q}}(t)$ and $\tilde{\tilde{\theta}}(t)$, $t \in [0, \tau]$, solve* (2.2) *and (3.6) given the boundary conditions $\tilde{\tilde{Q}}(0) = Q(0)$, $\tilde{\tilde{Q}}(\tau)$ satisfying (3.24), $\tilde{\tilde{\theta}}(\tau)$ satisfying*

$$\tilde{\tilde{\theta}}(\tau) = \tilde{\theta}(\tilde{\tilde{Q}}(\tau)) \tag{3.29}$$

and $\tilde{\tilde{\boldsymbol{K}}}(\tau) = \bar{\boldsymbol{K}}(\tau)$.

Under smoothness conditions (see Tsur and Zemel 2018), the $\tilde{\tilde{K}}_i(t)$ and $\tilde{\tilde{K}}_{ij}(t)$ processes are differentiable in time, hence can be viewed as driven by the well-defined MRAP investments

$$\tilde{\tilde{x}}_i(t) = d\tilde{\tilde{K}}_i(t)/dt + \delta\tilde{\tilde{K}}_i(t),\ i = n, r, d, s, \tag{3.30a}$$

$$\tilde{\tilde{x}}_{ij}(t) = d\tilde{\tilde{K}}_{ij}(t)/dt + \delta\tilde{\tilde{K}}_{ij}(t),\ i = n, r, d,\ j = D, I, A, E. \tag{3.30b}$$

The $\tilde{\tilde{x}}_i(t)$ and $\tilde{\tilde{x}}_{ij}(t)$ drive the MRAP capital stocks $\tilde{\tilde{K}}_i(t)$ and $\tilde{\tilde{K}}_{ij}(t)$ during $t \in [0, \tau]$ and satisfy

$$\tilde{\tilde{X}}(t) = \sum_{i=n,r,d,s} \tilde{\tilde{x}}_i(t) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} \tilde{\tilde{x}}_{ij}(t) = \bar{x},\ t \in [0, \tau). \tag{3.31}$$

We summarize the MRAP policy in:

Property 5 (MRAP)**.** *The MRAP policy consists of* $\{\tilde{\tilde{q}}(t), \tilde{\tilde{x}}(t), t \in [0, \tau]\}$ *and the associated state processes* $\{\tilde{\tilde{Q}}(t), \tilde{\tilde{K}}(t), t \in [0, \tau]\}$*, where* τ *and* $\{\tilde{\tilde{Q}}(t), \tilde{\tilde{q}}(t), \tilde{\tilde{K}}(t), t \in [0, \tau]\}$ *are characterized in Property 4, and* $\tilde{\tilde{x}}(t)$ *is defined in (3.30).*

We note that the length of the MRAP period, τ, is inversely related to the investment budget $\bar{x}$ and can be made arbitrarily short by appropriately increasing $\bar{x}$. We note also that such an increase in the investment budget can be temporary, as it need only apply during the MRAP (construction) period – up to time τ. From time τ onward, any investment budget $\bar{x}$ that allows the turnpike investment policy, i.e., that satisfies (3.19), suffices.

3.3 The optimal policy

Because the MRAP capital processes are subject to choice, the initial configuration $\tilde{\tilde{K}}(0)$ may differ from the actual initial capital stocks $K(0)$. However, (3.23) ensures that the total initial capital satisfies

$$\tilde{\tilde{\mathbf{K}}}(0) \leq \mathbf{K}(0),$$

where $\mathbf{K}(0)$ is the *actual* total capital stock. Suppose that at the initial time it is possible to reshuffle capital between the different stocks without changing the total capital stock. Then, it is possible to realize the initial capital configuration $\tilde{\tilde{K}}(0)$, under which the MRAP policy $\{\tilde{\tilde{q}}(t), \tilde{\tilde{x}}(t),\ t \in [0, \tau]\}$ is feasible. The optimal policy in this case is summarized in:

Property 6. *Under the "capital rearrangement" assumption, the optimal policy evolves along*

$$\{q^*(t), x^*(t)\} = \begin{cases} \{\tilde{\tilde{q}}(t), \tilde{\tilde{x}}(t)\}, & t \in [0, \tau) \\ \{\tilde{q}(t), \tilde{x}(t)\}, & t \geq \tau \end{cases} \tag{3.32}$$

and

$$\{Q^*(t), K^*(t)\} = \begin{cases} \{\tilde{\tilde{Q}}(t), \tilde{\tilde{K}}(t)\}, & t \in [0, \tau) \\ \{\tilde{Q}(t), \tilde{K}(t)\}, & t \geq \tau \end{cases}, \tag{3.33}$$

where the MRAP and turnpike policies are characterized in Properties 5 and 2, respectively.

We reiterate that characterizing the MRAP and turnpike policies requires solving single-state, dynamic optimization problems. The MRAP policy, summarized in Property 5, characterizes the construction of the water infrastructure, during which the investment budget is fully utilized. As soon as the water infrastructure reaches a level that allows implementing the turnpike policy, the optimal policy switches to the turnpike policy, characterized in Property 2, and evolves along it thereafter, eventually converging to the steady state specified in Property 1.

4 Regulation

As noted in the introduction, water allocation is rife with market failures, hence regulation is necessary for implementing the optimal policy, notwithstanding the use of market-based mechanisms in some segments of the water economy. In this section I discuss regulation based on volumetric pricing.[15]

It is important to distinguish between supply and demand regulation. The former entails managing the water supply $q_{ij}(t)$, $i = n, r, d$, $j = D, I, A, E$, from each source to each sector and the capital infrastructure needed to carry out these supplies. Demand regulation entails managing the sectoral demands $q_{\circ j}(t)$, $j = D, I, A, E$, and the capital infrastructure needed to distribute these allocations. The above characterization of the optimal policy provides

[15]On the various approaches commonly used in the regulation of water resources see Tsur and Dinar (1997), Johansson et al. (2002), Tsur et al. (2004) and references they cite.

straightforward rules for infrastructure regulation, namely a construction (MRAP) stage during which the water infrastructure is built until all capital stocks reach their turnpike trajectories and steering them along the turnpikes thereafter. As explained above, the MRAP stage is typically short and we focus on the turnpike stage. The reminder of this section focuses on implementing the turnpike water allocation $\tilde{q}(t) = \{\tilde{q}_{ij}(t),\ i = n, r, d,\ j = D, I, A, E\}$ via water pricing. The $\tilde{q}(t)$ trajectories identify the turnpike capital stocks, as specified in (3.8).

4.1 Water pricing

As noted above, Conditions (3.3)-(3.5) are of the form demand-equal-supply, with demands on the left-hand sides and (unit) supply costs on the right-hand sides, where the latter were denoted $\tilde{p}_{ij}(t)$, $i = n, r, d$, $j = D, I, A, E$, (cf. (3.9)-(3.11)). We denote by $p_{ij}(t)$ the associated unit cost in general (not necessarily along the turnpike). A pricing policy is a vector $P(t) = P_j(t), j = D, I, A, E$, that imposes the price $P_j(t)$ on sector j'th users at time t. Let $q(P(t)) = \{q_{ij}(P(t)), i = n, r, d, j = D, I, A, E\}$ represent the water allocation generated by the pricing policy $P(t)$. Then, $q(P(t))$ satisfies:

$$q_{ij}(P(t)) > 0 \text{ implies } p_{ij}(t) \leq P_j(t),\ i = n, r, d,\ j = D, I, A, E, \tag{4.1a}$$

and

$$D_j(q_{\circ j}(P(t)) \leq P_j(t) \text{ equality holding if } q_{\circ j}(t) > 0,\ \ j = D, I, A, E, \tag{4.1b}$$

where $q_{\circ j}(P(t))$ is defined in (2.1b). Condition (4.1a) states that water from source i is supplied to sector j only if the associated unit cost $p_{ij}(t)$ does not exceed the water price $P_j(t)$. Condition (4.1b) states that sector j'th users consume water along the demand curve, i.e., they demand water up to the point where their willingness to pay for an additional unit just equals the water price. If $q(P(t))$ is unique, we say that $P(t)$ implements the allocation $q(P(t))$. We seek the water prices that implement the turnpike allocation $\tilde{q}(t)$.

To that end, let

$$\tilde{P}_j(t) = \min_{i\in\{n,r,d\}} \tilde{p}_{ij}(t),\ j = D, I, A, E, \tag{4.2}$$

where the $\tilde{p}_{ij}(t)$'s are the unit costs along the turnpike, defined in (3.9)-(3.11).[16] Thus, $\tilde{P}_j(t)$ is the minimal unit cost of supplying water to sector j at time t along the turnpike policy. Then,

Property 7. *Along the turnpike policy, $\tilde{p}_{ij}(t) = \tilde{P}_j(t)$ for all sources i that supply water to sector j, i.e.,*

$$\tilde{q}_{ij}(t) > 0 \textit{ implies } \tilde{p}_{ij}(t) = \tilde{P}_j(t), i = n, r, d,\ j = D, I, A, E. \tag{4.3}$$

Proof. Conditions (3.3)-(3.5) imply

$$D_j(\tilde{q}_{\circ j}(t)) \le \tilde{P}_j(t) \text{ equality holding if } \tilde{q}_{\circ j}(t) > 0,\ j = D, I, A, E, \tag{4.4}$$

which together with (4.2) give (4.3). □

Property 7 states that, along the turnpike policy, the unit costs of allocating water to sector j are the same for all sources that supply water to this sector. Moreover, this unit cost is the minimal unit cost of supplying water to sector j from any feasible source $i = n, r, d$.

Let $\tilde{P}(t) = (\tilde{P}_D(t), \tilde{P}_I(t), \tilde{P}_A(t), \tilde{P}_E(t))$, where the $\tilde{P}_j(t)$'s are defined in (4.2). Then:

Property 8. *Suppose that the turnpike policy is unique (see conditions in the proof of Property 2). Then, $\tilde{P}(t)$ implements the turnpike allocation, i.e., $q(\tilde{P}(t)) = \tilde{q}(t)$.*

Proof. If the turnpike policy is unique, then $\tilde{q}(t) = \{\tilde{q}_{ij}(t),\ i = n, r, d,\ j = D, I, A, E\}$ is the unique solution of (3.3)-(3.5), given $\tilde{Q}(t)$, $\tilde{\theta}(t)$ and $\tilde{\xi}(t)$. But given $\tilde{p}(t)$, conditions (4.1) are equivalent to (4.3)-(4.4). The uniqueness of the turnpike allocation, then, implies $q(\tilde{P}(t)) = \tilde{q}(t))$. □

[16] If restriction (2.6) is imposed, then $\tilde{P}_D(t) = \min_{i\in\{n,d\}} \tilde{p}_{iD}(t)$.

In view of Property 8, $\tilde{P}(t)$ is referred to as the *turnpike* pricing policy. This policy implements the turnpike allocations $\tilde{q}(t)$, which in turn determine the turnpike capital stocks $\tilde{K}(t) = \{\tilde{K}_i(t), i = n, r, d, s, \tilde{K}_{ij}(t), i = n, r, d, j = D, I, A, E\}$ via (3.8). The supply cost at time t associated with the turnpike policy is therefore

$$C(\tilde{Q}(t), \tilde{q}(t)) + (\rho + \delta)\tilde{\mathbf{K}}(t), \tag{4.5}$$

where the variable cost $C(\tilde{Q}(t), \tilde{q}(t))$ is defined in (2.10) and $\tilde{\mathbf{K}}(t)$ is the total water infrastructure along the turnpike (cf. (3.2)).

The turnpike pricing policy raises the proceeds $\sum_{j=D,I,A,E} \tilde{q}_{\circ j}(t)\tilde{P}_j(t)$. If these proceeds suffice to cover the supply costs (4.5), we say that the policy is *self-sustained.* A self-sustained policy can be implemented without external financial intervention (such as subsidizing water suppliers), which greatly facilitates its implementation. We show that the turnpike pricing policy $\tilde{P}(t)$ is self-sustained.

4.2 Cost recovery

It is convenient to consider the case of linear variable cost functions:

$$C_i(q_{i\circ}) = c_i q_{i\circ},\ i = n, r, d, s, \;\; C_{ij}(q_{ij}) = c_{ij} q_{ij},\ i = n, r, d,\ j = D, I, A, E, \tag{4.6}$$

where c_n and $c_n(Q)$ are used interchangeably. If a policy is self-sustained in the linear case, it is also self-sustained for convex variable cost functions.

Under (4.6), the turnpike unit supply costs $\tilde{p}_{ij}(t)$ become

$$\tilde{p}_{nj}(t) = c_n(\tilde{Q}(t)) + c_{nj} + \mu_n + \mu_{nj} + \beta\left(c_s + \mu_s - \tilde{\xi}(t)\right) + \tilde{\theta}(t),\ j = D, I; \tag{4.7a}$$

$$\tilde{p}_{nj}(t) = c_n(\tilde{Q}(t)) + c_{nj} + \mu_n + \mu_{nj} + \tilde{\theta}(t),\ j = A, E; \tag{4.7b}$$

$$\tilde{p}_{rj}(t) = c_r + c_{rj} + \mu_r + \mu_{rj} + \beta\left(c_s + \mu_s\right) + \tilde{\xi}(t)(1 - \beta),\ j = D, I; \tag{4.8a}$$

$$\tilde{p}_{rj}(t) = c_r + c_{rj} + \mu_r + \mu_{rj} + \tilde{\xi}(t),\ j = A, E; \tag{4.8b}$$

$$\tilde{p}_{dj}(t) = c_d + c_{dj} + \mu_d + \mu_{dj} + \beta\left(c_s + \mu_s - \tilde{\xi}\right),\ j = D, I; \tag{4.9a}$$

$$\tilde{p}_{dj}(t) = c_d + c_{dj} + \mu_d + \mu_{dj},\ j = A, E. \tag{4.9b}$$

The turnpike pricing policy raises the proceeds $\sum_{j=D,I,A,E} \tilde{q}_{\circ j}(t)\tilde{P}_j(t)$ at time t, which using $\tilde{q}_{\circ j}(t) = \sum_{i=n,r,d} \tilde{q}_{ij}(t)$, can be expressed as

$$\sum_{j=D,I,A,E}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\tilde{P}_j(t).$$

Invoking Property 7 allows expressing these proceeds as

$$\sum_{j=D,I,A,E}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\tilde{p}_{ij}(t). \tag{4.10}$$

We now use (4.7)-(4.9) to evaluate (4.10). The domestic and industrial sectors ($j = D, I$) raise the proceeds

$$\sum_{j=D,I}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\tilde{p}_{ij}(t) = \sum_{j=D,I}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\left[c_i + c_{ij} + \mu_i + \mu_{ij} + \beta(c_s + \mu_s - \tilde{\xi}(t))\right] + \sum_{j=D,I} \tilde{q}_{nj}(t)\tilde{\theta}(t) + \sum_{D,I} \tilde{q}_{rj}(t)\tilde{\xi}(t).$$

Noting (2.1b) and (2.4), the terms involving β can be expressed as

$$\sum_{j=D,I}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\beta\left[c_s + \mu_s - \tilde{\xi}(t)\right] = \sum_{j=D,I} \tilde{q}_{\circ j}(t)\beta\left[c_s + \mu_s - \tilde{\xi}(t)\right] = \tilde{q}_{s\circ}(t)\left[c_s + \mu_s - \tilde{\xi}(t)\right].$$

The domestic and industrial proceeds, thus, become

$$\sum_{j=D,I}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\tilde{p}_{ij}(t) = \sum_{j=D,I}\ \sum_{i=n,r,d,s} \tilde{q}_{ij}(t)\left[c_i + \mu_i\right] + \sum_{j=D,I}\ \sum_{i=n,r,d} \tilde{q}_{ij}(t)\left[c_{ij} + \mu_{ij}\right] + \sum_{j=D,I} \tilde{q}_{nj}(t)\tilde{\theta}(t) + \sum_{j=D,I} \tilde{q}_{rj}(t)\tilde{\xi}(t) - \tilde{q}_{s\circ}(t)\tilde{\xi}(t),$$

where it is noted that the sum over i in the first term on the right-hand side includes the sewage s. Repeating these steps for the agriculture and environmental sectors ($j = A, E$), these sectors raise the proceeds

$$\sum_{j=A,E} \sum_{i=n,r,d} \tilde{q}_{ij}(t)\tilde{p}_{ij}(t) = \sum_{j=A,E} \sum_{i=n,r,d} \tilde{q}_{ij}(t)\,(c_i + \mu_i) + \sum_{j=A,E} \sum_{i=n,r,d} \tilde{q}_{ij}(t)\,(c_{ij} + \mu_{ij}) \\ + \sum_{j=A,E} \tilde{q}_{nj}(t)\tilde{\theta}(t) + \sum_{j=A,E} \tilde{q}_{rj}(t)\tilde{\xi}(t).$$

Summing the two expressions gives the total proceeds

$$\sum_{i=n,r,d,s} \tilde{q}_{i\circ}(t)\,[c_i + \mu_i] + \sum_{j=D,I,A,E} \sum_{i=n,r,d} \tilde{q}_{ij}(t)\,[c_{ij} + \mu_{ij}] + \sum_{j=D,I,A,E} \tilde{q}_{nj}(t)\tilde{\theta}(t) + \\ [\tilde{q}_{r\circ}(t) - \tilde{q}_{s\circ}(t)]\,\tilde{\xi}(t),$$

where (2.1a) was used to write $\tilde{q}_{r\circ}(t) = \sum_{j=D,I,A,E} \tilde{q}_{ij}(t)$. Noting (3.7a), the right-most term above vanishes. Moreover, (3.2) and (3.8) give

$$\tilde{q}_{i\circ}(t)\mu_i = (\rho + \delta)\tilde{K}_i(t),\ i = n, r, d, s, \quad \tilde{q}_{ij}(t)\mu_{ij} = (\rho + \delta)\tilde{K}_{ij}(t),\ i = n, r, d,\ j = D, I, A, E.$$

We thus conclude that

$$\sum_{j=D,I,A,E} \tilde{q}_{\circ j}(t)\tilde{P}_j(t) = C(\tilde{Q}(t), \tilde{q}(t)) + (\rho + \delta)\tilde{\mathbf{K}}(t) + \tilde{q}_{n\circ}(t)\tilde{\theta}(t), \tag{4.11}$$

implying that:

Property 9. *The proceeds $\sum_{j=D,I,A,E} \tilde{q}_{\circ j}(t)\tilde{P}_j(t)$ cover the supply costs (4.5) and leave the surplus $\tilde{q}_{n\circ}(t)\tilde{\theta}(t)$.*

The surplus proceeds, $\tilde{q}_{n\circ}(t)\tilde{\theta}(t)$, are associated with the shadow (or *in situ*) price of natural water $\tilde{\theta}(t)$. Sorting out the causes of this surplus requires understanding the role of the shadow price, which is best seen by considering the steady state $\hat{\theta}$, specified in (3.14):

$$\hat{\theta} = \frac{\hat{\vartheta} - c_n'(\hat{Q})R(\hat{Q}) + B_E^{is\prime}(\hat{Q})}{\rho - R'(\hat{Q})}.$$

The numerator represents the annual benefit obtained by a marginal change in the natural water stock. The denominator, $\rho - R'(\hat{Q})$, is the effective discount rate, consisting of the interest rate minus the marginal recharge. The division by $\rho - R'(\hat{Q})$, thus, gives the present value of the annual benefit stream associated with the marginal change in the natural water stock. We see that $\hat{\theta}$ consists of three terms. The scarcity component of $\hat{\theta}$ is represented by $\hat{\vartheta}$. Recall that $\vartheta(t)$ is the shadow price of the $Q(t) \geq \underline{Q}$ constraint. Thus, it vanishes if $\hat{Q} > \underline{Q}$, in which case the constraint is not binding and natural water is not scarce. If the constraint is binding, i.e., $\hat{Q} = \underline{Q}$, then the supply of natural water is limited by $\hat{q}_{n\circ} \leq R(\hat{Q})$ and $\hat{\vartheta}$ measures the price of this constraint, i.e., its effect on the optimal payoff (welfare). The term $-c_n'(\hat{Q}) \geq 0$ accounts for the effect of the natural water stock on the supply cost of natural water. Finally, the third term, $B_E^{is\prime}(\hat{Q})$, accounts for the instream value of natural water, embedded in its shadow (in situ) price. This term is an important component of the environmental price of water, to which we now turn.

4.3 Environmental water pricing

As discussed in Section 2, there are two types of environmental water: conveyed and instream. Conveyed environmental water, $q_{\circ E}(t) = \sum_{i=n,r,d} q_{iE}(t)$, originates from the water sources (natural, recycled and desalinated) and is conveyed to various sites for environmental purposes. The allocation $q_{iE}(t)$ entails the unit cost (variable plus capital) $c_i + c_{iE} + \mu_i + \mu_{iE}, i = n, r, d$. The demand for conveyed environmental water was denoted $D_E(\cdot)$ (see subsection 2.4) and defined in terms of the willingness to pay (WTP) for an additional unit of $q_{\circ E}$. The benefit generated by $q_{\circ E}$ is measured by the area underneath this demand curve, much like that of the other sectors (cf. (2.13)).

Unlike the other sectors, the q_{iE} is a public good, hence the consumption of individual users cannot be identified and users cannot be priced directly. As a result, the proceeds $\tilde{q}_{\circ E}(t)\tilde{P}_E(t)$ cannot be raised by charging users according to their water consumption and must be raised by other means, e.g., taxes. Nonetheless, the optimal (turnpike) allocations $\tilde{q}_{iE}(t), i = n, r, d$, can be calculated in the same way as those of the other sectors, based on the demand $D_E(\cdot)$ and unit supply costs $\tilde{p}_{iE}(t), i = n, r, d$.[17]

Instream water refers to natural water that could have been diverted or extracted but instead is left in its natural state (stream flows, aquifers, lakes) to support ecosystems and ecological amenities. The allocation of instream water entails no cost and its benefit is represented by the marginal instream benefit, $B_E^{is\prime}(\tilde{Q}(t))$, embedded in $\tilde{\theta}(t)$ (cf. (3.6)). Noting (4.7), the shadow price $\tilde{\theta}(t)$ is included in $\tilde{p}_{nj}(t), j = D, I, A, E$, hence also in the turnpike prices $\tilde{P}_j(t), j = D, I, A, E$, defined in (4.2). Thus, if the marginal instream value $B_E^{is\prime}(Q)$ is properly accounted for when calculating the shadow price $\tilde{\theta}(t)$, then this value is included in the turnpike prices $\tilde{P}(t)$ and the ensuing turnpike allocation properly accounts for the instream value of natural water.

The turnpike policy, thus, entails setting the prices $\tilde{P}_j(t), j = D, I, A$, on domestic, industry and agriculture users, and raising the proceeds $\sum_{j=D,I,A} \tilde{q}_{\circ j}(t)\tilde{P}_j(t)$, while centrally allocating the (conveyed) environmental water $\tilde{q}_{\circ E}(t)$ and collecting the ensuing proceeds $\tilde{q}_{\circ E}(t)\tilde{P}_E(t)$ via some indirect payments method (e.g., taxes). The allocation of instream water is the result of accounting for the marginal instream value $B_E^{is\prime}(\tilde{Q}(t))$ when calculating the shadow price $\tilde{\theta}(t)$ (cf. (3.6)).

We summarize the above discussion regarding pricing and allocation of environmental water in:

Property 10 (Environmental water). (*i*) *The demand for conveyed environmental water $D_E(q_{\circ E})$ is defined (like the other demands) as the WTP for an additional water unit at any feasible $q_{\circ E}$ allocation. Unlike the other sectors demands (which can be estimated by price-quantity data), the estimation of $D_E(\cdot)$ requires WTP valuation (based, e.g., on contingent*

[17] As noted in Section 2, the public good nature of $\tilde{q}_{\circ E}(t)$ implies that estimating the demand $D_E(\cdot)$ requires willingness to pay estimates (see Fleischer and Tsur 2009, Thiene and Tsur 2013, and references they cite).

valuation methods). Given $D_E(\cdot)$ and the supply cost data $\tilde{p}_{iE}(t), i = n, r, d$, the optimal allocation $\tilde{q}_{iE}(t)$ can be calculated together with the other $\tilde{q}_{ij}(t)$'s, as explained in Property 2. Due to the public good nature of $\tilde{q}_{\circ E}(t)$, its price $\tilde{P}_E(t)$, defined in (4.2) and calculated based on the $\tilde{p}_{iE}(t)$'s, cannot be used to implement the allocation $\tilde{q}_{\circ E}(t)$. The latter, thus, should be set by a water regulator and the ensuing proceeds $\tilde{q}_{\circ E}(t)\tilde{P}_E(t)$ should be raised by indirect means (e.g., taxes).

(ii) The allocation of instream water entails no cost and is driven by the marginal instream value of natural water, $B_E^{is\prime}(\tilde{Q}(t))$, embedded in the shadow price $\tilde{\theta}(t)$. The latter, in turn, affects the water price $\tilde{P}_n(t)$ (via its effect on the $\tilde{p}_{nj}(t)$'s), which determines $\tilde{q}_{n\circ}(t)$ and the ensuing $\tilde{Q}(t)$. When the marginal instream value $B_E^{is\prime}(\cdot)$ is properly accounted for when calculating $\tilde{\theta}(t)$, the ensuing allocation properly accounts for the value of instream water.

4.4 Timing and extent of desalination

Desalination is often more capital intensive than the supply of natural or recycled water, in which case $\gamma_d \ll \gamma_i$, $i = n, r$.[18] Thus, $\mu_d = (\rho + \delta)/\gamma_d \gg (\rho + \delta)\gamma_i = \mu_i$, $i = n, r$, implying that the capital cost component of desalination is larger than that of natural and recycled water. If desalinated water is demanded by sector j, i.e., $\tilde{q}_{dj}(t) > 0$, then $\tilde{p}_{dj}(t) = \tilde{p}_{ij}$ for all sources i from which sector j receives water (Property 7).

Suppose that natural water is supplied to all sectors, i.e., $\tilde{q}_{nj}(t) > 0$, $j = D, I, A, E$ – a common situation. Then, according to Property 8, sector j users demand desalinated water only if $\tilde{p}_{dj}(t) = \tilde{p}_{nj}(t)$. Comparing (3.9a) with (3.11a), retaining the linear costs (4.6), we see that for domestic users to demand desalinated water it must be that

$$c_n(\tilde{Q}(t)) + \mu_n + c_{nD} + \mu_{nD} + \tilde{\theta}(t) = c_d + \mu_d + c_{dD} + \mu_{dD}.$$

[18] Noting (2.7), supplying one water unit from source i requires $1/\gamma_i$ of capital K_i. If desalination is more capital intensive than recycled or natural water, it requires more capital to supply one unit of water, hence γ_d is smaller than both γ_n and γ_r.

Suppose that the distribution of natural and desalinated water to households is carried out jointly, so $c_{nD} = c_{dD}$ and $\mu_{nD} = \mu_{dD}$. In this case, the above condition becomes

$$c_n(\tilde{Q}(t)) + \mu_n + \tilde{\theta}(t) = c_d + \mu_d. \tag{4.12}$$

We thus conclude that:

Property 11. *When the distribution of natural and desalinated water to domestic users is carried out jointly (i.e., $c_{nD} = c_{dD}$ and $\mu_{nD} = \mu_{dD}$), Condition (4.12) is necessary for domestic users to demand desalinated water.*

The left-hand side of (4.12) is the unit cost of natural water (before distribution to the various sectors) and the right-hand side is the unit cost of desalination. The former evolves over time with the naturel water stock $\tilde{Q}(t)$ and the associated shadow price $\tilde{\theta}(t)$ while the latter is constant. Thus, Property 11 implies that as long as $c_n(\tilde{Q}(t)) + \mu_n + \tilde{\theta}(t) < c_d + \mu_d$, desalination is not desirable. If the supply of natural water exceeds natural recharge, i.e., $R(\tilde{Q}(t)) < \tilde{q}_{n\circ}(t)$, then $\tilde{Q}(t)$ decreases over time, implying that both $c_n(\tilde{Q}(t))$ and $\tilde{\theta}(t)$ increase (assuming $c_n'(Q) < 0$). Desalination becomes economically viable (if ever), according to Property 11, at the time the unit cost of natural water reaches the unit cost of desalination. From this time onward, the extent of desalination is determined by $\tilde{q}_{d\circ}(t)$. In the long run, desalination is justifiable if

$$c_d + \mu_d = c_n(\hat{Q}) + \mu_n + \hat{\theta}, \tag{4.13}$$

where $\hat{Q}$ and $\hat{\theta}$ are the steady state values of $\tilde{Q}(t)$ and $\tilde{\theta}(t)$ (see Property 1).

We summarize this discussion in:

Property 12. *When the distribution of natural and desalinated water to domestic users is carried out jointly (i.e., $c_{nD} = c_{dD}$ and $\mu_{nD} = \mu_{dD}$), desalination becomes economically viable (if ever) at the time condition (4.12) holds. From that time onward, the extent of desalination is determined by $\tilde{q}_{d\circ}(t)$. Condition (4.13) provides a convenient test for the desirability of desalination in the long run.*

4.5 Recap

The main findings of this section include:

(i) The turnpike pricing policy, which imposes the price $\tilde{P}_j(t)$ on sector j's users, implements the turnpike allocation $\tilde{q}(t) = \{\tilde{q}_{ij}, i = n, r, d, j = D, I, A, E\}$ and is self-sustained.

(ii) The proceeds raised by the turnpike pricing policy equal the supply costs plus $\tilde{q}_{n\circ}(t)\tilde{\theta}(t)$. The latter accounts for effects of the natural water stock on water scarcity, extraction cost and instream value.

(iii) Because individual consumption of conveyed environmental water, $\tilde{q}_{\circ E}(t)$, cannot be identified, this water cannot be priced by charging individual users. As a result, the allocation $\tilde{q}_{\circ E}(t)$ should be regulated (e.g., set by a regional water authority) and the associated proceeds, $\tilde{q}_{\circ E}(t)\tilde{P}_E(t)$, should be raised by indirect payment methods (e.g., taxes) that do not affect water demands.

(iv) The instream value of natural water is embedded in $\tilde{\theta}(t)$ and affects the allocation of natural water via the effect of $\tilde{\theta}(t)$ on the price of natural water $\tilde{P}_n(t)$.

(v) Condition (4.13) provides a convenient test for the desirability of desalination in the long run.

5 An example based on Israel's water economy

A description of Israel's water economy (sources, sectors, institutions) can be found in Kislev (2012). More up-to-date accounts of recycling and desalination, as well as allocation policies and prices, are discussed in Tsur (2015) and in ongoing publications of Israel's Water Authority.[19] Annual series of natural water supplies, dating back to 1932, are presented in

[19]The url address (in Hebrew) is http://www.water.gov.il/hebrew/pages/water-authority-info.aspx.

Weinberger et al. (2012). A popular account, together with an historical overview, can be found in Siegel (2015).

Israel's water law states that (translated from Hebrew): "The country's water resources are public property, controlled by the state and are designated for the needs of its residents and the development of the country. For the purpose of this law, water resources include: springs, streams, rivers, lakes, reservoirs, either surface or groundwater, natural or artificial, standing or flowing, including drainage water and sewage."[20] The responsibility for enforcing this law falls on the Water Authority (WA) and its various agencies. This responsibility includes: long run planning of the water economy, setting annual permits for extraction and diversion from natural sources, coordinating the construction of recycling facilities, managing tenders for desalination plants, and regulating water allocation to all sectors via an elaborate system of quotas (for agricultural users and environmental sites) and cost-based tariffs for all sectors (see discussion in Kislev 2012). All these tasks are addressed by the water economy model presented above.

In actual practice it may not be desirable to update water prices too often, due to high transactions costs or because users require stable prices to plan ahead (e.g., irrigators of perennial crops and contractors of treatment and desalination plants). In such cases, the water prices can be set at their steady state (long run) levels and transitory adjustments can be made by flexible quotas. I therefore confine attention to the steady state and calculate the turnpike prices $\hat{P}_j$, $j = D, I, A, E$, and the ensuing allocations $\hat{q} = \{\hat{q}_{ij}, i = n, r, d,\ j = D, I, A\}$. The allocations $\hat{q}$ identify the infrastructure $\hat{K} = \{\hat{K}_i, i = n, r, d, s;\ \hat{K}_{ij}, i = n, r, d,\ j = D, I, A\}$ via (3.17).

Table 1 presents parameters and functions. The 6.5 percent discount rate and the 3 percent depreciation rate are the return on capital and depreciation set by the Water Au-

[20] Israel's Water Law, 1959, Chapter 1 (http://water.gov.il/Hebrew/about-reshut-hamaim/Pages/Legislation.aspx).

thority (Belinkov 2014). The unit variable and capital costs, presented in Table 2, were calculated based on documented data.[21] The $\beta = 0.6$ share of sewage from the domestic and industrial allocations (see equation (2.4)) is taken from Tsur (2015). The linear demands, $D_j^{-1}(P_j) = a_j - b_j P_j$, are assumed for convenience and the $a_j, b_j, j = D, I, A$ parameters (presented in Table 3) were calibrated based on consumption-price data reported by the Water Authority.[22] The environmental demand parameters, a_E and b_E, and the marginal instream value, $B_E^{is\prime}(Q)$, are assumed.[23]

Table 1: Parameters and functions.

Parameter/function	Value	Description
ρ	0.065	discount rate
δ	0.03	depreciation rate
c_i, c_{ij}, μ_i, μ_{ij}	Table 2	unit supply costs (shekel per m^3)
β	0.6	sewage share from domestic and industrial sectors
$D_j^{-1}(P_j) = a_j - b_j P_j$	Table 3	demand coefficients
$B_E^{is\prime}(Q)$	0.01	marginal instream value (assumed)
$R(Q)$	$1,000$	natural water recharge (MCM/y)
$\underline{Q}$	0	lower bound on Q

The annual natural recharge of $1,000$ million m^3 (MCM/y) is based on the $1,125$ MCM/y average annual recharge during the period 1993-2009, reported in Weinberger et al. (2012, Table 7, p. 13). This figure excludes Gaza and the Eastern and Northeastern aquifers (underlying the West Bank). Subtracting the 100 MCM/y allocated to Jordan (under current

[21]These data were derived from the cost breakdown underlying the water charges determined by the Water Authority (see http://www.water.gov.il/Hebrew/Rates/Pages/Rates.aspx), from Belinkov (2014) and from conversations with Amir Shakarov of the Water Authority (whose help is gratefully acknowledged).

[22]The linear demand specification is made for illustration purpose. An empirical application, with detailed data and elaborate estimation of nonlinear demand specifications, is beyond the current scope. Water consumption data for the period 1998-2016 can be found in http://www.water.gov.il/Hebrew/ProfessionalInfoAndData/Allocation-Consumption-and-production/20164/thrich%20lfie%20matarot%201998-2016.pdf. Water tariffs for this period can be found in http://www.water.gov.il/Hebrew/Rates/Pages/prices-archive.aspx. The linear demand coefficients were calibrated as follows. First, consumptions are regressed on an intercept and prices (with a time trend if needed): $q_{\circ j}(t) = a_j - b_j P_j(t)$, $j = D, I, A$. The inverse demand functions are then $D_j = a_j/b_j - (1/b_j) q_{\circ j}$.

[23]Calibration requires willingness to pay estimates based on contingent valuation studies, which are presently not available.

agreements), leaves (after rounding) $1,000$ MCM/y. The zero lower bound $\underline{Q} = 0$ is a harmless normalization.

Table 2: Unit supply cost data in shekel per m^3 (the exchange rate at the time of writing is $\$1 = 3.7$ Israeli shekel).

	$(c_{ij},\, \mu_{ij})$				$(c_i,\, \mu_i)$
	Domestic	Industry	Ag	Env	
Natural	2.21, 1.63	2.10, 1.50	0.30, 0.50	0.30, 0.30	1.20, 1.00
Recycled		2.15, 1.50	0.40, 0.45	0.20, 0.45	0.30, 0.80
Desalinated	2.10, 1.74	2.10, 1.50	0.30, 0.80	0.30, 0.80	1.50, 1.30
Sewage					1.47, 1.16

Table 3: Demand coefficients.

	Domestic	Industry	Agriculture	Environment
a	1200	130	1200	500
b	35	5	130	40

The steady state allocation $\hat{q} = \{\hat{q}_{ij}, i = n, r, d, j = D, I, A, E\}$ and prices $\hat{P}_j$, $j = D, I, A, E$, are reported in Table 4. The desirable desalination capacity is 512 MCM/y, allocated to households (437 MCM/y) and industry (75 MCM/y). The current desalination capacity in Israel is 600 MCM/y. The larger capacity could be justified by expected increase in demand due to population growth. The table indicates that agricultural users should receive water mostly from recycling plants (559 MCM/y) and some from natural sources (172 MCM/y). In actual practice the allocation of natural water to agriculture is larger (see Tsur 2015). The reason could be insufficient infrastructure to convey recycled water from the densely populated center (where most recycling plants are located) to the heavily cultivated north and south. In other words, the recycling infrastructure is still in its construction (MRAP) stage. Indeed, observing the trends of water allocations during the past decade (see Tsur 2015), it is evident that the water economy evolves towards an allocation in which agriculture relies mostly on recycled water. Table 4 also reveals a substantial environmental water allocation of 364 MCM/y, supplied mostly from natural sources (317 MCM/y) and some from recycled sources (47 MCM/y). This allocation is at odds with the actual, much smaller, allocation. The reason for this divergence is twofold. First, the recommended allocation is based on assumed environmental water demand that may not reflect the true demand. Second, the infrastructure needed to convey environmental water (natural and recycled) to environmental sites may be underdeveloped (i.e., is still under construction).

Table 4: Steady state allocation (MCM/y) and water prices (shekel/m^3).

	$\hat{q}_{ij}$ (MCM/y)				$\hat{q}_{i\circ}$
	Domestic	Industry	Agriculture	Environment	
Natural (n)	509.45	1.66	171.97	316.92	1000
Recycled (r)	0	18.67	559.49	47.08	625.23
Desalinated (d)	437.54	74.73	0	0	512.28
$\hat{q}_{\circ j}$	946.99	95.06	731.46	364.00	
Sewage (s)	622.20	57.03			625.23
$\hat{P}_j$ (shekel per m^3)	7.23	6.99	3.60	3.40	
$\hat{\theta}$	0.60	(natural water shadow price)			
$\hat{\xi}$	1.65	(recycled water shadow price)			

Table 4 also presents the turnpike prices $\hat{P}_j, j = D, I, A, E$, that implement the (optimal) steady-state allocation $\hat{q}$, as well as the shadow prices $\hat{\theta}$ and $\hat{\xi}$. Recalling (4.7)-(4.9), $\hat{\theta}$ is included in the water price of natural water users, while $\hat{\xi}$ operates as a tax for users of recycled water and as a subsidy for domestic and industrial users (that contribute to the supply of recycled water via the sewage they generate). The two shadow prices are important components of the optimal prices and the ensuing water allocations: $\hat{\theta}$ accounts for the scarcity, extraction cost and instream value of natural water (in the present example, the extraction cost effect is assumed away by letting $c_n'(Q) = 0$); $\hat{\xi}$ is the shadow price of the $\hat{q}_{r\circ} \leq \hat{q}_{s\circ}$ constraint (i.e., that the supply of recycled water is limited by the sewage discharge). The shadow price $\hat{\theta}$ affect the allocation of natural water and the extent of desalination (Property 12), while the dual (tax/subsidy) role of $\hat{\xi}$ is vital in steering water allocation toward sewage generating users (households and industry) and in determining the use of recycled water.

Although the shadow price $\hat{\theta} = 0.6$ shekel/m^3 may seem small relative to the water prices, which range between 7.23 and 3.4 shekel/m^3 (Table 4), it has a pronounced effect on the desalination scale (of 512 MCM/y which is about 54 percent of domestic water consumption). This substantial desalination alleviates the water scarcity in two ways: first it increases the water input by augmenting nature as an external water source; second, each m^3 of desalinated water allocated to households and industrial users contributes an additional $\beta = 0.6\,m^3$ of recycled water (via the sewage these users generate). Thus, the shadow price $\hat{\theta}$ gives rise to the large scale desalination allocation and the latter, in turn, alleviates water scarcity and reduces the value of $\hat{\theta}$.[24]

I close this section by reiterating that the purpose of the above example is illustrative. A thorough application, based on which policy recommendations can be drawn, requires elaborate water demand estimation of all sectors as well as up-to-date supply costs data. It should allow for sub-sectors, such as agricultural users that cannot use recycled water (e.g., growers of field crops for direct human consumption) and those that can use water from all sources (e.g., growers of tree crops, cotton or animal feed). Such a division of the agriculture sector implies different prices for each of the subsectors, giving rise to different prices for recycled water and potable water (natural and desalinated) allocated to agriculture (as happens in actual practice). Additional sub-sources and sub-sectors, discussed in Section 2, can be incorporated when needed. The modular structure of the water economy framework facilitates such extensions.

[24]Indeed, calculating the steady state allocations and prices without desalination, i.e., assuming that no desalination plants were constructed, gives $\hat{\theta} = 3.28$ shekel/m^3.

6 Concluding comments

The water prices that implement the optimal water policy are derived. In addition to the usual variable and capital cost components, these prices contain two shadow price components: the *in situ* value of natural water and the scarcity price of recycled water. It is shown that the former accounts, in addition to the scarcity and extraction cost effects, to the *instream* value of natural water and has a pronounce effect on the onset and extant of desalination. The scarcity of recycled water stems from the fact that its supply is limited by the discharge of sewage from domestic and industrial users. The associated price (i.e., the shadow price of recycled water) is used as a subsidy for users that contribute to the supply of recycled water (i.e., domestic and industrial users) and as a tax for users that consume recycled water. The effect of this shadow price, therefore, extends beyond the allocation of recycled water and affects the allocation of water from all sources to all sectors.

The optimal pricing policy is shown to be self-sustained, in that the proceeds it generates cover the total supply costs. Actually, the proceeds exceed the supply costs by a surplus amount given by the proceeds associated with the natural water shadow price. This surplus is then traced to the components comprising the shadow price of natural water, namely scarcity, extraction cost and instream values. It is shown that the shadow price of natural water is central in determining the onset and extent of desalination.

The analysis distinguishes between two types of environmental water: conveyed and instream. The former refers to water conveyed from various sources to various environmental sites (e.g., to restore a polluted stream flow); the latter refers to leaving water, that otherwise could have been diverted or extracted, in its natural state (aquifer, lake, stream flow). Conveyed environmental water differs from water allocations to other sectors (domestic, industry and agriculture) in one important respect: the services it provides (environmental amenities) are public goods. As a result, individual users cannot be identified and priced according to their consumption, and this property bears implications for allocating this water and collection the ensuing proceeds. The allocation of instream water is determined by the shadow price of natural water when the latter properly accounts for the marginal benefit of instream water.

The modular structure of the water economy framework, presented in Section 2, allows adding sub-sources and sub-sectors in a straightforward fashion. Other extensions, such as growing water demands (due to population growth) or stochastic natural water recharge (due to stochastic precipitation) are welcome tasks for future research.

Appendix: Optimality proofs

The proofs follow the arguments in Tsur and Zemel (2018). The optimal policy consists of the feasible $q(t)$ and $x(t)$ processes that maximize (2.18) subject to (2.2) and (2.8), given $Q(0)$ and $K(0)$, where feasibility entails conditions (2.3), (2.5), (2.6), (2.7), (2.9) and nonnegativity of $q(t)$ and $x(t)$. This problem is referred to as the *full* problem, its solution is called the optimal policy and denoted with asterisk superscripts. The full problem is complicated because it contains the $K(t) = \{K_i(t), K_{ij}(t)\}$ as states, in addition to $Q(t)$. It greatly simplifies matter if the elements of $K(t)$ are considered as decisions rather than states. This is done twice: first, ignoring the investment budget constraint (2.9), and second accounting for this constraint. The former gives rise to the turnpike problem, and the latter to the restricted turnpike problem. The MRAP (or construction) policy coincides with the restricted turnpike policy when the investment constraint (2.9) is binding. It is then shown that the optimal policy begins with the MRAP policy until the turnpike policy becomes feasible, at which time the optimal policy switches to the turnpike policy and evolves along it thereafter.

A The turnpike policy

The turnpike policy treats the capital stocks $K(t)$ as decisions rather than states, hence consists of the feasible $q(t)$ and $K(t)$ processes that maximize (2.18) subject to (2.2) given $Q(0)$, where feasibility entails (2.3), (2.4), (2.5), (2.6), (2.7) and nonnegativity of $q(t)$ and $K(t)$ (i.e., the conditions of the full problem except (2.8) and (2.9)). Turnpike policy processes are denoted with a tilde overhead, e.g., $\tilde{q}(t)$ and $\tilde{K}(t)$. Under mild conditions (see Tsur and Zemel 2018), the elements of $\tilde{K}(t)$ are differentiable and give rise to the investments $\tilde{x}(t)$, specified in (3.18). Clearly, any $K(t)$ generated via the investments $x(t)$ that satisfy the budget constraint (2.9) is also feasible when the elements of $K(t)$ can be freely chosen but not vice versa. Thus,

Claim 2. *If the turnpike policy is feasible for the full problem, it must be optimal.*

The current-value Hamiltonian associated with the turnpike problem is

$$\mathcal{H}(t) = B(Q(t), q(t)) - C(Q(t), q(t)) - (\rho + \delta)\mathbf{K}(t) + \theta(t)[R(Q(t)) - q_{n\circ}(t)], \tag{A.1}$$

where $\theta(t)$ is the costate of $Q(t)$. The Lagrangian, which incorporates constraints (2.3), (2.5) and (2.7), is

$$\mathcal{L}(t) = \mathcal{H}(t) + \vartheta(t)\left(Q(t) - \underline{Q}\right) + \xi(t)\left(q_{s\circ}(t) - q_{r\circ}(t)\right) + \sum_{i=n,r,d,s} m_i(t)\left(\gamma_i K_i(t) - q_{i\circ}(t)\right) + \sum_{i=n,r,d}\ \sum_{j=D,I,A,E} m_{ij}(t)\left(\gamma_{ij} K_{ij}(t) - q_{ij}(t)\right), \tag{A.2}$$

where $\vartheta(t)$, $\xi(t)$ are the multipliers of (2.3), (2.5), and $m_i(t)$, $m_{ij}(t)$ are the multipliers corresponding to (2.7).

The necessary conditions associated with the $K_i(t)$ and $K_{ij}(t)$, recalling that they are subject to choice, i.e., $\partial\mathcal{L}(t)/\partial K_i(t) = \partial\mathcal{L}(t)/\partial K_{ij}(t) = 0$, give $m_i(t) = \mu_i$ and $m_{ij}(t) = \mu_{ij}$, where μ_i and μ_{ij} are defined in (3.2). Substituting μ_i and μ_{ij} for $m_i(t)$ and $m_{ij}(t)$ in (A.2), the necessary conditions associated with $q_{ij}(t)$ (i.e., $\partial\mathcal{L}/\partial q_{ij} = 0$) give (3.3)-(3.5) or their condensed form (3.12).

The necessary condition associated with the costate $\theta(t)$ is

$$\dot{\tilde{\theta}}(t) - \rho\tilde{\theta}(t) = c_n'(\tilde{Q}(t))\tilde{q}_{n\circ}(t) - B_E^{is}(\tilde{Q}(t)) - \tilde{\theta}(t)R'(\tilde{Q}(t)) - \tilde{\vartheta}(t),$$

verifying (3.6), where $C_n(Q, q_{n\circ}) = c_n(Q)q_{n\circ}$ is assumed. The complementary slackness conditions associated with (2.3) and (2.5) give (3.7), and the complementary slackness conditions associated with (2.7),

$$\mu_i[\gamma_i\tilde{K}_i(t) - \tilde{q}_{i\circ}(t)] = 0 \text{ and } \mu_{ij}[\gamma_{ij}\tilde{K}_{ij}(t) - \tilde{q}_{ij}(t)] = 0,$$

verify (3.8).

These conditions, together with (2.2), solve for the optimal $\tilde{Q}(t)$, $\tilde{\theta}(t)$, $\tilde{q}(t)$ and $\tilde{\xi}(t)$, given $\tilde{Q}(0) = Q(0)$ and the boundary values $\tilde{Q}(\infty) = \hat{Q}$ and $\tilde{\theta}(\infty) = \hat{\theta}$, where $\hat{Q}$ and $\hat{\theta}$ are specified in Property 1 and $\tilde{\vartheta}(t)$ satisfies (3.7b). This completes the proof of Property 2.

Notice that the turnpike problem involves a single state (the natural water stock $Q(t)$) and uniqueness of the turnpike policy is ensured when $\mathcal{H}(t)$ is strictly concave in $Q(t)$ and $q(t)$ (recall that $\mathcal{L}(t)$ is liner in the elements of $K(t)$). We assume that these conditions hold and the turnpike policy is unique. Under mild smoothness conditions, the $\tilde{K}_i(t)$'s and $\tilde{K}_{ij}(t)$'s are differentiable in time, hence can be viewed as driven by the turnpike investments, specified in (3.18). In view of Claim 2, we conclude that:

Claim 3. *If the turnpike policy is feasible, then* $(q^*(t), x^*(t)) = (\tilde{q}(t), \tilde{x}(t))$.

B The MRAP policy

Suppose the $q_{ij}(t)$ and $\{K_i(t), K_{ij}(t)\}$ are subject to choice, as in the turnpike problem, but the corresponding total capital $\mathbf{K}(t)$, defined in (2.17), cannot exceed the frontier process

$\bar{\mathbf{K}}(t) = (\bar{x}/\delta)\left(1 - e^{\delta t}\right) + \mathbf{K}(0)e^{-\delta t}$, defined in (3.22). Recall that $\bar{\mathbf{K}}(t)$ is the total capital process that departs from the *actual* initial total capital $\mathbf{K}(0)$ under the maximal total investment $X(t) = \bar{x}$. We call this problem the *restricted turnpike* problem, the associated optimal processes are called *restricted turnpike* processes and denoted with a double-tilde overhead, e.g., $\tilde{\tilde{Q}}(t)$, $\tilde{\tilde{q}}(t)$, $\tilde{\tilde{K}}(t)$.

The restricted turnpike policy consists of the feasible $q(t)$ and $K(t)$ that maximize (2.18) subject to (2.2) given $Q(0)$, where feasibility entails (2.3), (2.4), (2.5), (2.6), (2.7), non-negativity of $q(t)$ and $K(t)$, and (3.23), i.e., the feasibility conditions of the turnpike problem plus restriction (3.23).The Hamiltonian of the restricted turnpike problem is the same as the Hamiltonian of the turnpike problem, defined in (A.1), and the Lagrangian equals $\mathcal{L}(t) + \eta(t)[\bar{\mathbf{K}}(t) - \mathbf{K}(t)]$, where $\mathcal{L}(t)$ is the Lagrangian of the turnpike problem, defined in (A.2), and $\eta(t) \geq 0$ is the shadow price (multiplier) of (3.23).

The necessary conditions with respect to $K_i(t)$ and $K_{ij}(t)$ give (3.25). Substituting the $m_i(t)$ and $m_{ij}(t)$ in $\mathcal{L}(t)$, the necessary conditions with respect to $q_{ij}(t)$ give (3.26). The necessary condition associated with the costate $\theta(t)$ gives (3.6) with $\tilde{\tilde{\theta}}(t)$ replacing $\tilde{\theta}(t)$ and the complementary slackness conditions (3.7) now hold with $\tilde{\tilde{\vartheta}}(t)$ and $\tilde{\tilde{\xi}}(t)$ replacing $\tilde{\vartheta}(t)$ and $\tilde{\xi}(t)$, respectively. The complementary slackness conditions associated with (2.7) give (3.28). Finally, the complementary slackness conditions associated with (3.23) is

$$\eta(t)[\bar{\mathbf{K}}(t) - \tilde{\tilde{\mathbf{K}}}(t)] = 0, t \in [0, \tau]. \tag{B.1}$$

Claim 4. *Suppose $\tau > 0$, so $\bar{\boldsymbol{K}}(t) < \tilde{\boldsymbol{K}}(\tilde{\tilde{Q}}(t))$ during $t \in [0, \tau)$ while the turnpike policy is not feasible. Then, $\tilde{\tilde{\boldsymbol{K}}}(t)$ satisfies (3.27).*

Proof. Suppose $\tilde{\tilde{\mathbf{K}}}(t_0) < \bar{\mathbf{K}}(t_0)$ at some time $t_0 < \tau$. Then, $\tilde{\tilde{\mathbf{K}}}(t) < \bar{\mathbf{K}}(t)$ for all $t \in [t_0, \tau]$. This is so because the frontier process $\bar{\mathbf{K}}(t)$ is the result of investing at the maximal rate $\bar{x}$, hence cannot be overtaken from below by a feasible total capital process that departed from the same initial total capital $\mathbf{K}(0)$. Condition (B.1), then, implies that $\eta(t) = 0$ for all

$t \geq t_0$. If follows that conditions (3.26) coincide with conditions (3.12) for $t \geq t_0$. But the necessary conditions of the turnpike and the restricted turnpike problems differ only due to the difference between (3.26) and (3.12) and these two sets of conditions differ only when $\eta(t) > 0$. Thus, if $\eta(t) = 0$ for $t \geq t_0$, the turnpike policy that departs from $\tilde{\tilde{Q}}(t_0)$ satisfies all the necessary conditions of the restricted turnpike problem but is infeasible because the (unique) turnpike policy is not feasible prior to time τ – a contradiction. We conclude that $\eta(t) > 0$ during $t \in [0, \tau)$, implying, noting (B.1), that $\tilde{\tilde{\mathbf{K}}}(t) = \bar{\mathbf{K}}(t)$ during $t \in [0, \tau]$, verifying the claim. $\square$

The above discussion completes the proof of Property 3 and Part (i) of Property 4. In view of Property 3 and Claim 4, the restricted turnpike problem can be reformulated as: find the feasible $q(t), K(t)$ and τ that maximize

$$\int_0^\tau \left[B(q(t)) - C(Q(t), q(t)) - (\rho + \delta)\mathbf{K}(t)\right] e^{-\rho t} dt + e^{-\rho\tau}\tilde{v}(Q(\tau)) \tag{B.2}$$

subject to (2.2) and

$$\mathbf{K}(t) = \bar{\mathbf{K}}(t),\ t \in [0, \tau] \tag{B.3}$$

given $Q(0)$, where feasibility entails the feasibility conditions of the turnpike problem and $\tilde{v}(Q)$ is the value (optimal payoff) associated with the turnpike policy that departs from the initial natural water stock Q. The Hamiltonian and Lagrangian associated with this problem are the same as those of the restricted turnpike problem and the necessary conditions include those of the restricted turnpike problem plus the transversality condition associated with the choice of τ:

$$B(\tilde{\tilde{q}}(\tau)) - C(\tilde{\tilde{Q}}(\tau), \tilde{\tilde{q}}(\tau)) - (\rho + \delta)\bar{\mathbf{K}}(\tau) + \tilde{\tilde{\theta}}(\tau)\left(R(\tilde{\tilde{Q}}(\tau)) - \tilde{\tilde{q}}_{n\circ}(\tau)\right) = \rho\tilde{v}(\tilde{\tilde{Q}}(\tau)). \tag{B.4}$$

Following Dynamic Programming arguments, the right-hand side above is expressed as

$$\rho\tilde{v}(\tilde{\tilde{Q}}(\tau)) = B(\tilde{q}(\tilde{\tilde{Q}}(\tau)) - C(\tilde{\tilde{Q}}(\tau), \tilde{q}(\tilde{\tilde{Q}}(\tau)) - (\rho+\delta)\tilde{\mathbf{K}}(\tilde{\tilde{Q}}(\tau)) + \tilde{\theta}(\tilde{\tilde{Q}}(\tau))\left(R(\tilde{\tilde{Q}}(\tau)) - \tilde{q}_{n\circ}(\tilde{\tilde{Q}}(\tau))\right). \tag{B.5}$$

Now, $\tilde{\tilde{q}}(\tau) = \tilde{q}(\tilde{\tilde{Q}}(\tau))$ because at time τ the (unique) turnpike policy becomes feasible, hence also optimal, for the restricted turnpike problem (Claim 1). Conditions (3.8), (3.28) and (B.3), then, imply $\tilde{\mathbf{K}}(\tilde{\tilde{Q}}(\tau)) = \tilde{\tilde{\mathbf{K}}}(\tau) = \bar{\mathbf{K}}(\tau)$, verifying (3.24). Invoking $\tilde{\tilde{q}}(\tau) = \tilde{q}(\tilde{\tilde{Q}}(\tau))$, (3.24), (B.4) and (B.5) verifies (3.29), completing the proof of Property 4.

C The optimal policy

The *full* problem is a restricted version of the *restricted turnpike* problem. This is so because, while the two problems share the same objective and constraints, the capital stocks $K(t) = \{K_i(t), K_{ij}(t)\}$ are states driven by investments under the full problem, whereas under the restricted turnpike problem they are subject to choice. Thus,

Claim 5. *If the restricted turnpike policy is feasible for the full problem, it must be optimal.*

The question, then, is whether the restricted turnpike policy is feasible for the full problem. The answer is in the affirmative only if $\tilde{\tilde{K}}(0) \leq K(0)$, i.e., when the initial capital stocks under the restricted turnpike policy do not exceed the *actual* initial stocks. But the elements of $\tilde{\tilde{K}}(0)$ are subject to choice and the elements of $K(0)$ are exogenously given, hence in general the two differ. However, the corresponding *total* capital stocks satisfy

$$\tilde{\tilde{\mathbf{K}}}(0) \leq \mathbf{K}(0), \tag{C.1}$$

because the restricted turnpike policy satisfies (3.23) at all times and particularly at the initial time. We thus conclude that:

Claim 6. *Suppose that at the initial time, given the actual total capital $\boldsymbol{K}(0)$, the existing capital can be reshuffled between the different stocks, such that any capital configuration $K(0) = \{K_i(0), K_{ij}(0)\}$, satisfying*

$$\sum_{i=n,r,d,s} K_i(0) + \sum_{i=n,r,d} \sum_{j=D,I,A,E} K_{ij}(0) \leq \boldsymbol{K}(0), \tag{C.2}$$

is feasible. Then, the restricted turnpike policy is feasible for the full problem.

Claims 5 and 6, together with the smoothness conditions ensuring the existence of $\tilde{\tilde{x}}(t)$ and $\tilde{x}(t)$ (see Tsur and Zemel 2018), complete the proof of Property 6.

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