האוניברסיטה העברית בירושלים

The Hebrew University of Jerusalem

המרכז למחקר בכלכלה חקלאית

The Center for Agricultural Economic Research

המחלקה לכלכלת סביבה וניהול

Department of Environmental Economics and Management

Discussion Paper No. 1.17

Water policy guidelines: A comprehensive approach

by

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September 27, 2017

Abstract

We study water management in the context of a prototypical water economy containing the main water sources and user sectors. A water policy consists of water allocation from each source to each user sector at each point of time as well as the capital investments needed to carry out these allocations. We show that the optimal policy brings the water capital stocks (infrastructure and equipment) to well-specified turnpike processes as rapidly as possible and evolves along these turnpikes thereafter, eventually converging to a unique steady state. Implications for water pricing are discussed.

Keywords: Water economy; intertemporal management, scarcity, turnpike, most rapid approach.

JEL classification: C61, Q25, Q28

*Helpful comments and discussions by Ariel Dinar are gratefully acknowledged.

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1 Introduction

We study the salient features of water management in the context of a prototypical water economy, consisting of the main water sources and user sectors. The term "water economy" refers to a collection of water sources and user sectors that are entwined via physical (equipment, infrastructure) and social (institutions, norms, laws) capital. Water economies vary in both respects and their idiosyncratic features affect the range of feasible policies (see the examples in Saleth and Dinar 2004, Tsur et al. 2004). Without committing to a particular setting, we characterize the optimal water policy in terms of intertemporal water allocations from each source to any user sector and the investments in (physical) capital needed to carry out these allocations.

We find that the optimal policy proceeds along two stages: a most-rapid-approach (MRAP) stage followed by a turnpike (singular) stage. In the first stage, the capital stocks (equipment, infrastructure) are driven as rapidly as possible (i.e., at the maximal feasible rate) to well-specified turnpike trajectories. During the second stage, the capital stocks evolve at a more moderate rate along their turnpikes, eventually converging to a steady state. The duration of the MRAP stage is inversely related to the (overall) investment budget and can be made arbitrarily short. Thus, most of the process evolution takes place along the turnpikes and specifying the optimal water policy, therefore, involves mainly specifying the turnpike policy. This simplifies the water management task considerably, as the turnpike policy includes only the water stock as a state variable but not the capital stocks.

The primary source of water is nature (rainfall, lakes, stream flows, aquifers). In regions where the (sustainable) supply of natural water suffices to meet human and environment needs, water is not scarce and managing it may not be high on the priority list. Such regions decrease in number over time due to demographic and climatic trends. In many populated regions, water scarcity has become critical (see Dinar and Tsur 2015), stressing the need for proper management.

Two sources of produced water can be added to natural sources: recycling and desalination. Recycled water is the outcome of collecting and treating domestic and industrial sewage. As such, its supply is determined by the allocation of water to these sectors. Sewage treatment is required primarily due to health and environmental considerations, disregarding whether the treated water is reused later on. The level of treatment (secondary, tertiary) determines the range of feasible uses of the recycled water. These considerations bear important implications for the allocation of water in general as well as for the level of treatment and who should pay for the different stages of the recycling process. The model developed herein accounts for these considerations.

Desalinization is, for all practical purposes, an unlimited source of water, hence can be considered as a backstop technology. However, at the current state of technology, it is an expensive source. This raises the issues of when to begin desalination (if at all) and the extent of desalination over time. The framework developed herein addresses these concerns.

The present effort builds on Tsur's (2009) framework and extends it in a number of ways. While Tsur (2009) simplified the dynamic aspects by considering steady states, the water policy characterized herein is fully intertemporal, covering both the water allocation from each source to each user sector at each point of time and the gradual capital investment (equipments, infrastructure) needed to carry out these allocations.

The next section specifies the stylized water economy that will serve as a basis for the analysis and defines feasible water policies in this economy. The optimal policy is shown, in Section 3, to evolve along the two aforementioned stages and to eventually converge to a unique steady state. Section 4 concludes and an appendix contains technical details and proofs.

2 The water economy

The water economy specified in Tsur (2009) provides a convenient starting point. Water is derived from three main sources and is allocated to four main user sectors. While the primary source of water is nature (rainfall, aquifers, lakes, reservoirs, stream flows), water can be derived also from recycling facilities and from desalination plants. The four main user sectors are domestic (residential), agriculture (irrigators), industry and the environment.[1] We use the index $i = n, r, d$, to denote natural (n), recycling (r) and desalination (d) sources, and the index $j = D, A, I, E$, to signify domestic (D), agriculture (A), industry (I) and environment (E) sectors.

We denote by $q_{ij}(t)$ the supply flow (say, million cubic meter per year) from source i to sector j in year t. The annual water supply from source i is

$$q_{i\circ}(t) = \sum_{j=A,D,I,E} q_{ij}(t),\ i = n, d, r, \tag{2.1a}$$

and the annual allocation to sector j is

$$q_{\circ j}(t) = \sum_{i=n,r,d} q_{ij}(t),\ j = D, A, I, E. \tag{2.1b}$$

Water sources

We discuss the three water sources in turn.

[1]Focusing on water scarce regions, we ignore hydropower and navigation sectors.

Natural: Natural water is mostly derived from a finite, replenishable stock $Q(t) \in [0, \bar{Q}]$, which evolves over time according to

$$\dot{Q}(t) = R(Q(t)) - q_{n\circ}(t), \tag{2.2}$$

where $R(\cdot)$ is a decreasing and concave recharge function and the upper bound $\bar{Q}$ satisfies $R(\bar{Q}) = 0$.[2] The lower bound

$$Q(t) \geq 0 \tag{2.3}$$

implies that the supply of natural water cannot exceed $R(0)$ when $Q(t) = 0$ (the zero lower bound is a standard normalization). The capital (infrastructure, equipment) needed to allocate (pump, treat, convey, distribute) natural water is denoted K_n.

Recycled water: Recycled water is derived from treated domestic and industrial sewage. Let $q_{s\circ}(t)$ denote the flow of domestic and industrial sewage at time t. Then,

$$q_{s\circ}(t) = \beta(q_{\circ D}(t) + q_{\circ I}(t)), \tag{2.4}$$

where $\beta \leq 1$ accounts for water consumption and loss during sewage collection and treatment.[3] The capital employed in sewage collection and treatment is denoted K_s.

The share of the treated sewage that is reused constitutes the supply of recycled water $q_{r\circ}(t)$. Thus,

$$q_{r\circ}(t) \leq q_{s\circ}(t). \tag{2.5}$$

[2] Allowing for multiple natural stocks, each with its own recharge process, is outlined in Tsur (2016). If irrigation and environmental water contribute to the recharge of underlying aquifers, the recharge function takes the form $R(Q, q_{\circ A}, q_{\circ E})$, where R decreases in Q and increases in both $q_{\circ A}$ and $q_{\circ E}$. In the interest of simplicity, the latter effects are ignored.

[3] Under current technology and practice, $\beta \approx 0.65$ (see Tsur 2015).

While getting rid of the treated sewage may be costless, reusing it may require further treatment and conveyance facility (pipelines, pumps) to convey the treated water from the recycling plants to potential users. The capital needed by these additional recycling activities is denoted K_r.

The distinction between $q_{s\circ}$ and $q_{r\circ}$ is needed because sewage collection and treatment, on the one hand, and the allocation of the treated water to potential users, on the other hand, are two separate activities. The former is (often) required by health and environmental regulation, disregarding whether the treated water is reused later on. Reusing the treated water, on the other hand, is a policy decision that depends on the cost of conveying the recycled water from the treatment facilities to potential users and on the demand for the recycled water. The treatment level (secondary, tertiary) entails restrictions on potential uses of the recycled water. For example, secondary-treated water may not be allowed to irrigate certain crops and health regulations may prohibit the allocation of any recycled water to households, i.e.,

$$q_{rD}(t) = 0. \tag{2.6}$$

Desalination: The supply of desalinated water at time t, $q_{d\circ}(t)$, is restricted only by the capacity of existing desalination plants, i.e., by the available desalination capital, denoted K_d.

Supply cost

The cost of water supply includes variable and fixed costs. The former entails costs of variable inputs, such as labor, energy and material; the latter includes mainly the cost of capital. Both of these components vary spatially and temporally (see examples in Renzetti 1999, Harou et al. 2009, Allen et al.

2014).

Capital (fixed) cost: Water supply from source i at time (year) t, denoted $q_{i\circ}(t)$, is restricted by source i's capital stock, $K_i(t)$, according to

$$q_{i\circ}(t) \leq \gamma_i K_i(t),\ i = n, s, r, d, \tag{2.7}$$

where γ_i is a capital utilization parameter, indicating the maximal annual supply of water from source i per unit K_i. The latter evolves in time according to

$$\dot{K}_i(t) = x_i(t) - \delta K_i(t),\ \ i = n, s, r, d, \tag{2.8}$$

where $x_i(t)$ represents investment rate in K_i at time t and δ is a constant depreciation rate (assumed equal for all capital stocks). If a total investment budget $\bar{x}$ is imposed on the water economy, then

$$\sum_i x_i(t) \leq \bar{x} \tag{2.9}$$

(unless otherwise indicated, $\sum_i$ is short-hand for $\sum_{i=n,s,r,d}$).

Let $\mathbf{K}(t) = \sum_i K_i(t)$ denote the total water capital stock, which, noting (2.8), evolves in time according to

$$\dot{\mathbf{K}}(t) = \sum_i x_i(t) - \delta \mathbf{K}(t). \tag{2.10}$$

Let $\bar{\mathbf{K}}(t)$ represent the solution of (2.10) when $\sum_i x_i = \bar{x}$ and $\bar{\mathbf{K}}(0) = \mathbf{K}(0) = \sum_i K_i(0)$, i.e.,

$$\dot{\bar{\mathbf{K}}}(t) = \bar{x} - \delta \bar{\mathbf{K}}(t), \tag{2.11}$$

which gives

$$\bar{\mathbf{K}}(t) = \frac{\bar{x}}{\delta}\left(1 - e^{-\delta t}\right) + \mathbf{K}(0)e^{-\delta t}. \tag{2.12}$$

Clearly, any $\mathbf{K}(t) = \sum_i K_i(t)$ trajectory satisfying (2.9) also satisfies

$$\sum_i K_i(t) \leq \bar{\mathbf{K}}(t), \tag{2.13}$$

but the converse may not hold. This is so because the capital constraint (2.13) is weaker than the investment constraint (2.9) in that the former allows for temporary violation of the latter if at other times (2.9) holds as a strong inequality such that (2.13) holds at all times. This distinction will prove useful in characterizing the optimal policy.

Variable costs: The (annual) variable cost of supplying $q_{i\circ}$ is represented by the increasing and convex functions $C_i(q_{i\circ})$, $i = s, r, d$. For $i = n$ (natural water), C_n may depend also on the stock of natural water Q, in which case $C_n(Q, q_{n\circ})$ is non-increasing and concave in Q and increasing and convex in $q_{n\circ}$, e.g., $C_n(Q, q_{n\circ}) = C_n(Q)q_{n\circ}$, where the unit extraction cost function $C_n(Q)$ is non-increasing and convex. These functions account for the costs of variable inputs such as temporary labor, energy and materials.

Extensions allowing for source-and-sector specific costs, e.g., when water allocated to households requires extra treatment or when water distribution entails sector specific costs, are discussed in Tsur (2016).

Water sectors: demand and surplus

Sector j's annual (inverse) demand for water is denoted $D_j(q_{\circ j})$. This curve measures the quantity of water demanded by sector j at any water price and can be interpreted as the price sector j's users are willing to pay for the last (marginal) unit of water.[4]

[4]There is a large literature on sectoral water demands. Examples of agricultural water demand include Just et al. (1983), Moore et al. (1994), Howitt (1995), Mundlak (2001),

The annual gross surplus of sector j generated by $q_{\circ j}$ (before subtracting the cost of water supply) is the area underneath the demand curve to the left of $q_{\circ j}$:

$$B_j(q_{\circ j}) = \int_0^{q_{\circ j}} D_j(s)ds, \tag{2.14}$$

Subtracting the variable costs of supply and the investment expenditures gives the net (annual) benefit flow at time t:

$$\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - \sum_i x_i(t).$$

(As a matter of notation, unless otherwise indicated, $\sum_j$ and $\sum_i$ indicate summation over $j = D, A, I, E$, and $i = n, s, r, d$, respectively.)

Water policy and welfare

A policy consists of the water allocation $q(t) \equiv \{q_{ij}(t),\, i = n, r, d;\, j = A, D, I, E\}$ and investment rates $x(t) \equiv \{x_i(t),\, i = n, s, r, d\}$ throughout the indefinite planning horizon $t \geq 0$, where $q_{n\circ}(t) = \sum_j q_{nj}(t)$ determines $Q(t)$ via (2.2) and $x_i(t)$ determines $K_i(t)$ via (2.8). A water policy generates the payoff (welfare)

$$\int_0^\infty \left(\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - \sum_i x_i(t) \right) e^{-\rho t} dt,$$

where ρ is the time rate of discount. Using (2.8) to eliminate the investment rate $x_i(t)$ and integrating by parts the resulting $\dot{K}_i(t)$ terms yields an equiva-

Tsur et al. (2004), Schoengold et al. (2006), Scheierling et al. (2006); examples of urban and industrial demands include Baumann et al. (1997), Renzetti (2002, 2015), Worthington and Hoffmann (2006), Olmstead et al. (2007), House-Peters and Chang (2011), Baerenklau et al. (2014), Smith and Zhao (2015); examples of environmental water demand include Dudley and Scott (1997), Loomis et al. (2000), Pimentel et al. (2004), Thiene and Tsur (2013), Koundouri and Davila (2015).

lent form for the payoff

$$\int_0^\infty \left(\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - (\rho+\delta)\mathbf{K}(t) \right) e^{-\rho t} dt + \mathbf{K}(0),$$

where it is recalled that $\mathbf{K}(t) = \sum_i K_i(t)$ and $\mathbf{K}(0)$ is the initial (total) capital stock. Subtracting the initial capital, i.e., purchasing it at the outset, gives the payoff

$$\int_0^\infty \left(\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - (\rho+\delta)\mathbf{K}(t) \right) e^{-\rho t} dt. \tag{2.15}$$

In this form, the annual cost of $\mathbf{K}(t)$ is $(\rho+\delta)\mathbf{K}(t)$, which accounts for the interest payments on a $\mathbf{K}$-worth loan ($\rho\mathbf{K}$) plus depreciation cost ($\delta\mathbf{K}$).

We close this section with a list of properties satisfied by the functions comprising the above water economy, to be used in the subsequent analysis.

Assumption 1. *(i) $R(\cdot)$ is decreasing and concave;*
(ii) $B_j(\cdot)$, $j = D, I, A, E$, are increasing and strictly concave (follows from (2.14) and the property that the $D_j(\cdot)$'s are decreasing);
(iii) $C_n(Q, q_{n\circ})$ is non-increasing and convex in Q, increasing and convex in $q_{n\circ}$;
(iv) $C_i(q_{i\circ})$, $j = s, r, d$, are increasing and convex;
(v) All the above functions are twice continuously differentiable.

3 Optimal policy

The optimal policy is the feasible $\{q(t), x(t), t \geq 0\}$ that maximizes (2.15) subject to the state dynamics (2.2) and (2.8), given the initial natural water stock $Q(0)$ and capital stocks $K(0) = (K_n(0), K_s(0), K_r(0), K_d(0))'$. A policy is feasible if it satisfies (2.5), (2.6), (2.7), (2.13) and nonnegativity of $q(t)$, $K(t)$ and $Q(t)$, where $q_{s\circ}(t)$ is defined in (2.4). As presented here, the problem is formulated in terms of 5 state variables (Q and K_i, $i = n, s, r, d,$) hence is hard to solve directly. We carry out this task by relating the optimal processes to the corresponding processes obtained for similar problems that differ in that the stocks K_i are treated as decisions rather than as states, hence are free of the state dynamic constraint (2.8). In one version of these problems we also relax the capital constraint (2.13) while in another version this constraint is imposed. These problems are simpler to solve, and their respective processes allow a complete characterization of the optimal water policy.

We find that the optimal policy is to drive the capital stocks $K_i(t)$ to well-specified turnpike processes, denoted $\tilde{K}_i(t)$, and maintain them along these turnpikes thereafter. During the approach to the turnpike, the investment budget (2.9) is fully utilized, hence the optimal policy is akin to the so-called most-rapid-approach (MRAP). The turnpikes $\tilde{K}_i(t)$ are the capital stocks that would be chosen had the K_i's been freely determined. In actual practice the K_i's cannot be freely chosen, but rather follow the state dynamic constraint (2.8) subject to (2.13). The optimal policy, it turns out, is to bring the $K_i(t)$'s as rapidly as possible to the desired, turnpike processes. Similar policies, which approach a moving target (process) as rapidly as possible, have been studied in Tsur and Zemel (2000) who referred to them as Non-Standard Most Rapid Approach Paths (NSMRAP), generalizing the MRAP to a fixed target introduced by Spence and Starrett (1975).

We begin by specifying the turnpike policy and the associated trajectories. We then characterize how the optimal trajectories approach their turnpike counterparts by showing that they evolve along trajectories that are optimal to a problem referred to as the *auxiliary* problem. The auxiliary problem is similar to the turnpike problem in that it treats the $K_i(t)$'s as decisions (rather than states) but differs in that it imposes the capital constraint (2.13). As long as the frontier capital stock $\bar{\mathbf{K}}(t)$ lies below the total turnpike capital stock, i.e, $\bar{\mathbf{K}}(t) < \sum_i \tilde{K}_i(t)$, the turnpike policy is not feasible. Requiring that the investment budget $\bar{x}$ is large enough ensures that $\bar{\mathbf{K}}(t)$ increases fast enough and eventually (at a finite time) reaches the total turnpike capital stock, at which time the turnpike policy becomes feasible to the auxiliary problem. We show that until that time the optimal capital trajectories evolve along $\bar{\mathbf{K}}(t)$, by fully utilizing the investment budget $\bar{x}$. As soon as the turnpike policy becomes feasible, the optimal policy switches to it, allowing a more moderate capital growth. Of all the $K_i(t)$ trajectories satisfying (2.13), those evolving along $\bar{\mathbf{K}}(t)$ approach the turnpike policy most rapidly (i.e., at the shortest time). The optimal policy is therefore a non-standard most rapid approach (NSMRAP) to the turnpike policy, justifying the use of the term "turnpike".

3.1 The turnpike policy

Suppose that the capital services of K_i can be rented (annually) at the unit price $\rho + \delta$, instead of being developed gradually according to (2.8).[5] In this case, the capital stocks K_i (rather than the investment rates x_i) are decision (control) variables, leaving the natural water stock $Q(t)$ as the sole state of the problem. We refer to this problem as the turnpike problem, use the modifier 'turnpike' to any of the associated optimal processes, and denote them by the overhead tilde " ˜ " symbol.

[5]This would be the case if investments are unconstrained, so the water infrastructure can adjust instantly. Notice that $\rho + \delta$ is the competitive rental rate of a capital stock that depreciates at the rate δ when the market interest rate is ρ.

The turnpike policy consists of the feasible $q(t) = \{q_{ij}(t),\ i = n,r,d,\ j = D,I,A,E\}$ and $K(t) = (K_n(t), K_s(t), K_r(t), K_d(t))'$, $t \geq 0$, that maximize (2.15) subject to (2.2) given $Q(0)$, where feasibility entails (2.3), (2.5), (2.6), (2.7) and nonnegativity of $q(t)$ and $K(t)$. As it is single-state, infinite horizon and autonomous, the turnpike policy can readily be characterized. With $\theta(t)$ denoting the costate of $Q(t)$, the (current-value) Hamiltonian associated with this problem is

$$\mathcal{H}(t) = \sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - (\rho+\delta)\sum_i K_i(t) + \theta(t)[R(Q(t)) - q_{n\circ}(t)]$$

and the Lagrangian is

$$\mathcal{L}(t) = \mathcal{H}(t) + \vartheta(t)Q(t) + \xi(t)[q_{s\circ}(t) - q_{r\circ}(t)] + \sum_i \mu_i(t)\left(\gamma_i K_i(t) - q_{i\circ}(t)\right),$$

where $\vartheta(t)$, $\xi(t)$ and $\mu_i(t)$ are the Lagrange multipliers of (2.3), (2.5) and (2.7), respectively, $q_{i\circ}(t)$ and $q_{\circ j}(t)$ are defined in (2.1) and $q_{s\circ}(t)$ is defined in (2.4).

The necessary conditions associated with the choice of $K_i(t) \geq 0$ are

$$\partial\mathcal{L}/\partial K_i = -(\rho+\delta) + \gamma_i\mu_i(t) \leq 0, \tag{3.1}$$

equality holding if $K_i(t) > 0$, $i = n,s,r,d$, in which case the condition gives

$$\tilde{\mu}_i = (\rho+\delta)/\gamma_i,\ i = n,s,r,d, \tag{3.2}$$

as the shadow prices of the capital capacity constraints (2.7).

With $c_i(\cdot) \equiv \partial C_i(\cdot)/\partial q_{i\circ}$, $i = n,s,r,d$, denoting the marginal supply costs, the necessary conditions associated with the $q_{ij}(t)$'s are:

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_n(\tilde{Q}(t), \tilde{q}_{n\circ}(t)) + \tilde{\mu}_n + \tilde{\theta}(t) + \beta[c_s(\tilde{q}_{s\circ}(t)) + \tilde{\mu}_s] - \beta\tilde{\xi}(t), \quad (nDI)$$

equality holding if $\tilde{q}_{nj}(t) > 0$, $j = D, I$;

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_n(\tilde{Q}(t), \tilde{q}_{n\circ}(t)) + \tilde{\mu}_n + \tilde{\theta}(t), \tag{nAE}$$

equality holding if $\tilde{q}_{nj}(t) > 0$, $j = A, E$;

$$D_I(\tilde{q}_{\circ I}(t)) \leq c_r(\tilde{q}_{r\circ}(t)) + \tilde{\mu}_r + \beta[c_s(\tilde{q}_{s\circ}(t)) + \tilde{\mu}_s] + \tilde{\xi}(t)(1-\beta), \tag{rI}$$

equality holding if $\tilde{q}_{rI}(t) > 0$;

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_r(\tilde{q}_{r\circ}(t)) + \tilde{\mu}_r + \tilde{\xi}(t), \tag{rAE}$$

equality holding if $\tilde{q}_{rj}(t) > 0$, $j = A, E$;

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_d(\tilde{q}_{d\circ}(t)) + \tilde{\mu}_d + \beta[c_s(\tilde{q}_{s\circ}(t)) + \tilde{\mu}_s] - \beta\tilde{\xi}(t), \tag{dDI}$$

equality holding if $\tilde{q}_{dj}(t) > 0$, $j = D, I$;

$$D_j(\tilde{q}_{\circ j}(t)) \leq c_d(\tilde{q}_{d\circ}(t)) + \tilde{\mu}_d, \tag{dAE}$$

equality holding if $\tilde{q}_{dj}(t) > 0$, $j = A, E$.

The costate $\tilde{\theta}(t)$ evolves in time according to

$$\dot{\tilde{\theta}}(t)) - \rho\tilde{\theta}(t) = C_{nQ}(\tilde{Q}(t), \tilde{q}_{n\circ}(t)) - \tilde{\theta}(t)R'(\tilde{Q}(t)) - \tilde{\vartheta}(t), \tag{3.6}$$

where $C_{nQ}(\tilde{Q}(t), \tilde{q}_{n\circ}(t)) \equiv \partial C_n/\partial Q$ and $R' \equiv \partial R/\partial Q$. Finally, the complementary slackness conditions are

$$\tilde{\xi}(t)[\tilde{q}_{s\circ}(t) - \tilde{q}_{r\circ}(t)] = 0, \tag{3.7a}$$

$$\tilde{\vartheta}(t)\tilde{Q}(t) = 0 \tag{3.7b}$$

and

$$\tilde{\mu}_i[\gamma_i\tilde{K}_i(t) - \tilde{q}_{i\circ}(t)] = 0. \tag{3.7c}$$

Conditions (3.2) and (3.7c) give

$$\tilde{K}_i(t) = \tilde{q}_{i\circ}(t)/\gamma_i,\ i = n, s, r, d, \tag{3.8}$$

Notice that (3.8), which specifies the relation between the turnpike capital stocks and the water allocation policy, holds also when $\tilde{K}_i(t) = 0$.

We use $\tilde{v}(Q)$ to denote the turnpike value function, i.e., the payoff under the turnpike policy given $Q(0) = Q$.

The water allocation conditions (nDI)-(dAE) are of the form demand-equals-supply, with demand on the left-hand sides and unit supply cost on the right-hand sides. As such, the latter can be interpreted as water prices. To better see this interpretation, suppose the marginal costs c_i, $i = n, s, r, d$, are independent of the supply flows $\tilde{q}_{i\circ}(t)$, $i = n, s, r, d$. The price of natural water includes $c_n(\tilde{Q}(t)) + \tilde{\mu}_n + \tilde{\theta}$, appearing on the right-hand sides of (nDI) and (nAE). The first term is the marginal cost component of the water price, the second term is the capital cost component and the third term is the scarcity component. The marginal cost component raises the proceeds $c_n(\tilde{Q}(t))\tilde{q}_{n\circ}(t)$, which (when $C_n(Q, q_{n\circ}) = c_n(Q)q_{n\circ}$) exactly cover the variable cost of supply. The proceeds raised by the capital cost component, $\tilde{\mu}_n$, equal (noting (3.2)) $\tilde{q}_{n\circ}(t)\tilde{\mu}_n = (\rho+\delta)\tilde{K}_n(t)$, which exactly cover the annual cost of $\tilde{K}_n$. The water proceeds raised by the scarcity component, $\tilde{\theta}(t)\tilde{q}_{n\circ}(t)$, have no contemporary cost counterpart but rather future costs that will be borne by future users due to higher supply cost (if C_n decreases with Q) or lack of sufficient natural water (if Q will be depleted, following which natural water supply cannot exceed $R(0)$).

The term $\beta(c_s + \tilde{\mu}_s)$, included in the water allocation conditions of the domestic and industrial sectors (the right-hand sides of conditions (nDI), (rI)

and (dDI)), is associated with the cost of collecting and treating sewage. When included as a component in the water prices facing the domestic and industrial sectors, the proceeds raised by this term cover the cost of sewage collection and treatment. To see this, recall that each cubic meter of water allocated to either the domestic or industrial sectors generates the share β of sewage that must be collected and treated. Allocating $\tilde{q}_{\circ D} + \tilde{q}_{\circ I}$ generates the (annual) sewage flow $\beta(\tilde{q}_{\circ D} + \tilde{q}_{\circ I})$.

The term $\beta\tilde{\mu}_s$ generates the proceeds $\beta\tilde{\mu}_s[\tilde{q}_{\circ D}(t) + \tilde{q}_{\circ I}(t)]$, which, noting (2.4), equals $\tilde{\mu}_s\tilde{q}_{s\circ}(t)$. The latter, noting (3.2), equals $(\rho+\delta)\tilde{K}_s(t)$, or the annual cost of the sewage capital, when $\tilde{K}_s(t) = \tilde{q}_{s\circ}(t)/\gamma_s$. Likewise, the proceeds raised by the term βc_s cover the variable cost $c_s(\tilde{q}_{s\circ})\tilde{q}_{s\circ}(t)$ of sewage collection and treatment (and equal the variable cost exactly when c_s is independent of $q_{s\circ}$). Notice that only users that generate sewage (i.e., domestic and industrial users) are required to pay for sewage, as the $\beta(c_s + \tilde{\mu}_s)$ term appears only on the right-hand sides of the conditions determining water allocation to these sectors.

The term $c_r + \tilde{\mu}_r$, included in the price of recycled water (cf. conditions (rI) and (rAE)), is associated with the variable and capital (fixed) cost of recycling. Likewise, the term $c_d + \tilde{\mu}_d$, included in the price of desalinated water (cf. conditions (dDI) and (dAE)), accounts for the variable and capital cost of desalination. When $C_i(q_{i\circ}) = c_i q_{i\circ}$, the water proceeds raised by $c_i, i = r, d$, equal the variable costs $c_i\tilde{q}_{i\circ}, i = r, d$, and it is easy to show, as shown above, that the water proceeds raised by $\tilde{\mu}_i, i = r, d$, just suffice to cover the capital cost $(\rho+\delta)\tilde{K}_i, i = r, d$.

Note that desalination is often more capital intensive than recycling or natural water supply, in which case $\gamma_d \ll \gamma_i$, $i = n, r$.[6] This implies $\tilde{\mu}_d = (\rho + \delta)/\gamma_d \gg (\rho + \delta)\gamma_i = \tilde{\mu}_i$, $i = n, r$, so the capital cost component of K_d exceeds that of K_n and K_r. Thus, the price of desalinated water, which include $\tilde{\mu}_d = (\rho + \delta)/\gamma_d$, is often higher than that of natural or recycled water. If $D_j(0) \leq c_d + \tilde{\mu}_d$, $j = A, E$, agricultural and environmental users are not willing to pay the desalination price and condition (dAE) implies $\tilde{q}_{dA} = \tilde{q}_{dE} = 0$. If the same holds also for the domestic and industrial sectors, i.e., $D_j(0) \leq c_d + \tilde{\mu}_d + \beta(c_s + \tilde{\mu}_s) - \beta\tilde{\xi}$, $j = D, I$, then desalination is not desirable at all.

The term $\tilde{\xi}(t)$, appearing on the water allocation conditions of sectors that generate sewage or consume recycled water, is the shadow price of constraint (2.5) and represents the scarcity price of recycled water. This term vanishes if the constraint is not binding but otherwise can be positive. As such, it acts as a subsidy to water allocations that increase the supply of sewage, thereby relaxing this constraint, i.e., natural and desalinated water allocated to the domestic and industrial sectors (cf. conditions (nDI) and (dDI)). This subsidy encourages reallocation of natural water away from agriculture (or environmental) uses to domestic (or industry) users, because the latter receives one cubic meter for each reallocated cubic meter whereas agricultural growers loose only the fraction $1-\beta$ since they get back the share β in the form of recycled water. With $\beta \approx 0.65$ (see Tsur 2015), the effect of $\tilde{\xi}(t)$ can have far reaching consequences. The $\tilde{\xi}(t)(1 - \beta)$ term in condition (rI) accounts for the dual role of the industrial sector vis-à-vis recycled water: the scarcity price $\tilde{\xi}(t)$ is due to the consumptive role whereas the subsidy $-\beta\tilde{\xi}(t)$ is due to

[6] Recall that $\gamma_i K_i$ is source i's supply constraint. If desalination is more capital intensive it requires more capital to supply the same annual flow, hence γ_d is smaller than both γ_n and γ_r.

its contributing role, as a share β of the industrial water allocation is returned to the recycling facility in the form of sewage. The use of $\tilde{\xi}(t)$ to subsidize domestic and industrial users (which contribute to sewage and to the supply of recycled water) is cost-neutral, as it is fully paid for by the $\tilde{\xi}(t)$ component in the price of recycled water.

The turnpike capital stocks as states driven by investments

Because the turnpike problem is infinite horizon and autonomous, the optimal $\tilde{q}_{ij}(t)$ processes can be represented as functions of the state $Q(t)$ (see, e.g., Leonard and Long 1992):

$$\tilde{q}_{ij}(t) = f_{ij}(Q(t)),\ i = n, r, d;\ \ j = D, A, I, E. \tag{3.9}$$

Noting (3.8), the $\tilde{K}_i(t)$)'s can be expressed as

$$\tilde{K}_i(t) = f_{i\circ}(Q(t))/\gamma_i,\ i = n, s, r, d, \tag{3.10}$$

where $f_{i\circ}(Q) = \sum_{j=D,I,A,E} f_{ij}(Q)$ and $f_{\circ j}(Q) = \sum_{i=n,s,r,d} f_{ij}(Q)$.

Noting (3.2), equation (3.10) implies $(\rho + \delta)\tilde{K}_i(t) = \tilde{\mu}_i f_{i\circ}(Q(t))$. The turnpike value (the payoff under the turnpike policy) can thus be expressed as

$$\tilde{v}(Q) = \int_0^\infty [\sum_j B_j(f_{\circ j}(Q(t))) - C_n(Q(t), f_{n\circ}(Q(t)) - \sum_{i=s,r,d} C_i(f_{i\circ}(Q(t))) \\ - \sum_i \tilde{\mu}_i f_{i\circ}(Q(t))]e^{-\rho t}dt. \tag{3.11}$$

Assuming that the turnpike value is differentiable (see conditions in Benveniste and Scheinkman 1979), the $f_{ij}(\cdot)$'s are differentiable as well, implying that $\dot{\tilde{K}}_i(t) = f'_{i\circ}(Q(t))\dot{Q}(t)/\gamma_i$, $i = n, s, r, d$, exist. It follows that

$$\tilde{x}_i(t) = \dot{\tilde{K}}_i(t) + \delta\tilde{K}_i(t),\ i = n, s, r, d, \tag{3.12}$$

are well defined and can be interpreted, noting (2.8), as the turnpike investment rates driving the $\tilde{K}_i(t)$'s.

We assume that the investment budget constraint $\bar{x}$ is large enough to support these turnpike investment rates at all times:

$$\bar{x} > \sum_i \tilde{x}_i(t) \ \forall \ t \geq 0. \tag{3.13}$$

In particular, this assumption ensures that if the turnpike stocks $\tilde{K}_i(t_0)$, $i = n, s, r, d$, satisfy the total capital constraint (2.13) at some time $t_0 \geq 0$, they will continue to do so for all $t > t_0$.

Convergence to a steady state

The curvature properties in Assumption 1 ensure that the turnpike trajectories $\tilde{q}(t)$ and $\tilde{Q}(t)$ are unique.[7] Moreover, because the turnpike problem is infinite horizon, autonomous and involves a single bounded state, $\tilde{Q}(t)$ converges to a steady state (Tsur and Zemel 2014), where $\tilde{Q}$, $\tilde{K}_i$, $\tilde{q}_{ij}$, $\tilde{\theta}$, $\tilde{\vartheta}$ and $\tilde{\xi}$ remain constant. We denote steady state values by a hat " ˆ " overhead.

From (2.2)

$$R(\hat{Q}) = \hat{q}_{n\circ} \tag{3.14}$$

and (3.6) gives, using (3.14),

$$\hat{\theta} = \frac{\hat{\vartheta} - C_{nQ}(\hat{Q}, R(\hat{Q}))}{\rho - R'(\hat{Q})}, \tag{3.15}$$

where $\hat{\vartheta}$ satisfies (cf. (3.7b))

$$\hat{\vartheta}\hat{Q} = 0. \tag{3.16}$$

Noting (3.7a), $\hat{\xi}$ satisfies

$$\hat{\xi}[\hat{q}_{s\circ} - \hat{q}_{r\circ}] = 0. \tag{3.17}$$

[7] Substituting $\sum_i \tilde{\mu}_i q_{i\circ}(t)$ for $(\rho + \delta)\sum_i K_i(t)$, the Hamiltonian of the turnpike problem is strictly concave in (Q, q), ensuring uniqueness of the turnpike policy.

Equations (nDI)-(dAE) and (3.14)-(3.17) provide 15 conditions to solve for the $\hat{q}_{ij}$ (of which there are 11 free variables when (2.6) is imposed), $\hat{Q}$, $\hat{\theta}$, $\hat{\xi}$ and $\hat{\vartheta}$ as follows: if equations (nDI)-(dAE), (3.14)-(3.17) admit nonnegative solution $\hat{q}_{ij}$, $\hat{Q}$, $\hat{\theta}$, $\hat{\xi}$ with $\hat{\vartheta} = 0$, then these are the steady states values; if no such (non-negative) solutions exist, then $\hat{Q} = 0$ and $\hat{\vartheta} \geq 0$ is set in order to satisfy (nDI)-(dAE), (3.14)-(3.17). Noting (3.8), the steady state values of the capital stocks K_i are

$$\hat{K}_i = \hat{q}_{i\circ}/\gamma_i, \; i = n, s, r, d. \tag{3.18}$$

We summarize the above discussion in:

Proposition 1. *The turnpike trajectories are unique and converge to the unique steady states specified by (nDI)-(dAE) and (3.14)-(3.17) from any initial* $Q(0) \in [0, \bar{Q}]$.

3.2 The turnpike policy vis-à-vis the optimal policy

As a matter of notation, the problem of choosing the feasible $\{q(t), x(t)\}$ policy that maximizes the welfare (2.15) is referred to as the "full problem." The solution of the full problem is called the optimal policy and is indicated by the asterisk " * " superscript. As before, the tilde " ˜ " overhead signifies evaluation under the turnpike policy.

Property 1. *Suppose that* $K_i^*(t_0) = \tilde{K}_i(t_0)$, $i = n, s, r, d$, *at some time* $t_0 \geq 0$. *Then, from time* t_0 *onward,* $(q^*(t), x^*(t)) = (\tilde{q}(t), \tilde{x}(t))$, $Q^*(t) = \tilde{Q}(t)$ *and* $K_i^*(t) = \tilde{q}_{i\circ}(t)/\gamma_i$, $i = n, s, r, d$, *where* $\tilde{q}(t)$ *and* $\tilde{Q}(t)$ *are the optimal processes corresponding to* $\tilde{v}(Q^*(t_0))$, *and* $\tilde{x}(t)$ *is defined in (3.12).*

Proof. Without loss of generality, set $t_0 = 0$. The turnpike problem is less restricted than the full problem because in the former problem K_i can be freely chosen at any point of time, whereas in the latter problem they are state variables, driven by investments according to (2.8) and subject to the capital constraint (2.13). It follows that every policy that is feasible for the full problem is also feasible for the turnpike problem.

The converse, however, is not necessarily true because arbitrary K-processes which are admissible under the turnpike problem may fail to satisfy (2.13). This would be the case if the initial capital stocks are small, so that the initial total capital $\bar{\mathbf{K}}(0)$ falls below $\sum_i \tilde{K}_i(0)$. In contrast, if $K_i^*(0) = \tilde{K}_i(Q^*(0))$, $i = n, s, r, d$, condition (2.13) is satisfied from the outset, and assumption (3.13) ensures that it will continue to hold at all subsequent times. Thus, the turnpike policy $\{\tilde{q}(t), \tilde{x}(t)\}$ is feasible for the full problem at all times. Moreover, this policy is optimal for the full problem because a policy yielding a higher value would give a higher value also for the turnpike problem, contradicting the characterization of the turnpike policy as optimal for the latter problem. □

The above Property implies that

$$v(Q, K) \leq v(Q, \tilde{K}(Q)). \tag{3.19}$$

This is so because $v(Q, \tilde{K}(Q)) = \tilde{v}(Q)$ and the turnpike value cannot fall short of the value of the full problem, given the same initial natural water stock Q.

Property 1 states that if the turnpike policy is feasible, it must be optimal. It remains to characterize the optimal policy when the turnpike policy is not feasible. To that end, the following auxiliary problem will prove useful.

3.3 An auxiliary problem

Consider the problem of finding the feasible $q(t) = \{q_{ij}(t), i = n, r, d,\ j = D, I, A, E\}$ and $K(t) = (K_n(t), K_s(t), K_r(t), K_d(t))'$ that maximize

$$\int_0^\infty \left(\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - (\rho+\delta) \sum_i K_i(t) \right) e^{-\rho t} dt \tag{3.20}$$

subject to (2.2) and (2.11) given $Q(0)$ and $\bar{\mathbf{K}}(0)$, where the feasibility constraint are the same as those of the full problem, including the capital constraint (2.13). This problem is referred to as the auxiliary problem and its optimal policy and trajectories are identified by the modifier "auxiliary" and the double-tilde "$\tilde{\tilde{}}$" overhead.

The auxiliary problem is similar to the turnpike problem in that it treats the $K_i(t)$'s (together with the $q_{ij}(t)$'s) as decision variables, but differs in that it imposes the capital constraint (2.13), which requires introducing the capital frontier process $\bar{\mathbf{K}}(t)$ as a second state variable (in addition to $Q(t)$). Notice that $\bar{\mathbf{K}}(t)$ is exogenous, as its time evolution, which depends on the initial capital stocks, is fully specified in (2.12). The auxiliary problem is thus a restricted version of the turnpike problem. Therefore, if the turnpike policy is feasible for the auxiliary problem, it must also be optimal for the auxiliary problem.

Now, the turnpike policy is feasible for the auxiliary problem when $\bar{\mathbf{K}}(t) \geq \sum_i \tilde{K}_i(t)$.[8] Consequently, let τ be the first time this condition is satisfied, i.e.,

$$\tau = \min\{t \geq 0 | \bar{\mathbf{K}}(t) \geq \sum_i \tilde{K}_i(t)\}. \tag{3.21}$$

[8]Recall that $\bar{\mathbf{K}}(t)$ initiates from $\bar{\mathbf{K}}(0) = \sum_i K_i(0)$, as specified in (2.12), and the $\tilde{K}_i(t)$ processes are the optimal capital stocks corresponding to $\tilde{v}(Q(0))$. Thus, there is no ambiguity regarding the time argument in $\bar{\mathbf{K}}(t)$ and in the $\tilde{K}_i(t)$'s.

Clearly, $\tau > 0$ when $\bar{\mathbf{K}}(0) < \sum_i \tilde{K}_i(0)$ and $\tau = 0$ when $\bar{\mathbf{K}}(0) \geq \sum_i \tilde{K}_i(0)$. Moreover, because assumption (3.13) ensures that $\bar{\mathbf{K}}(t)$ grows faster than $\sum_i \tilde{K}_i(t)$, τ must be finite and from time τ onward the turnpike policy is feasible for the auxiliary problem. We conclude that:

Property 2. *The turnpike policy is optimal for the auxiliary problem from time τ onward.*

Comparing the auxiliary problem with the full problem, it is seen that the latter is a restricted version of the former. This so because, while both problems impose constraint (2.13), the auxiliary problem treats the capital stocks – the K_i's – as decision variables whereas the full problem treats them as state variables driven by investments. It therefore follows that:

Property 3. *If the auxiliary policy is feasible for the full problem, then it is also optimal.*

A question then arises regarding when, or under what conditions, the auxiliary policy is feasible for the full problem. The following characterization of the auxiliary policy helps clarifying this issue.

Property 4 (Characterization of the auxiliary policy)**.** *The auxiliary processes satisfy:*

$$\tilde{\tilde{K}}_i(t) = \tilde{\tilde{q}}_{i\circ}(t)/\gamma_i, \; i = n, s, r, d; \tag{3.22}$$

$$\sum_i \tilde{\tilde{K}}_i(t) = \bar{\mathbf{K}}(t) \; for \, t \leq \tau; \tag{3.23}$$

The $\tilde{\tilde{K}}_i(t)$'s are differentiable in time over $t \in [0, \tau]$, hence

$$\tilde{\tilde{x}}_i(t) = \dot{\tilde{\tilde{K}}}_i(t) + \delta \tilde{\tilde{K}}_i(t), \; i = n, s, r, d, \; t \in [0, \tau] \tag{3.24}$$

are well defined and (recalling (2.8)) can be interpreted as the investment rates driving the $\tilde{\tilde{K}}_i(t)$*'s during* $t \in [0, \tau]$.

Thus, although the auxiliary problem involves no investments, as the K_i's are determined by choice, the $\tilde{\tilde{K}}_i(t)$'s evolve smoothly in time and could have been driven by the smooth investments $\tilde{\tilde{x}}_i(t)$, defined in (3.24). The proof of Property 4 is presented in the appendix.

Noting (3.23), the auxiliary capital processes, defined in (3.22), satisfy the capital constraint (2.13), hence Property 4 identifies the condition ensuring that the auxiliary policy is feasible for the full problem from the outset, namely:

Property 5. *If* $K(0) = \tilde{\tilde{K}}(0)$*, then the auxiliary policy is feasible (hence optimal) for the full problem with* $\tilde{\tilde{K}}_i(t)$ *driven by the investments* $\tilde{\tilde{x}}_i(t)$, $i = n, s, r, d$*, specified in (3.24).*

3.4 The optimal policy

Property 3 states that if the auxiliary policy is feasible for the full problem then it is also optimal. Property 4 shows that the auxiliary stock processes, the $\tilde{\tilde{K}}_i(t)$'s, evolve smoothly in time and identifies the auxiliary investment processes that could have driven them. Property 5 then identifies the condition under which the auxiliary policy is feasible from the outset. The three properties imply:

Property 6. *If* $K(0) = \tilde{\tilde{K}}(0)$*, then the optimal policy is characterized by*

$$(q^*(t), x^*(t), Q^*(t), K^*(t)) = \begin{cases} (\tilde{\tilde{q}}(t), \tilde{\tilde{x}}(t), \tilde{\tilde{Q}}(t), \tilde{\tilde{K}}(t)), & t \leq \tau \\ (\tilde{q}(t), \tilde{x}(t), \tilde{Q}(t), \tilde{K}(t)), & t > \tau \end{cases}. \tag{3.25}$$

Notice from equation (3.23) that during $t \in [0, \tau]$, $\sum_i \tilde{\tilde{K}}_i(t) = \bar{\mathbf{K}}(t)$, implying that $\sum_i \tilde{\tilde{x}}_i(t) = \bar{x}$, so the investment budget $\bar{x}$ is fully utilized. Thus, the turnpikes are approached as rapidly as possible. A special case occurs when the water system is built from scratch, i.e., $K(0) = 0$, in which case $\bar{\mathbf{K}}(0) = 0$, implying that $\tilde{\tilde{K}}(0) = 0$ as well. Thus,

Property 7. *If $K(0) = 0$, the condition $K(0) = \tilde{\tilde{K}}(0)$ is trivially satisfied and the optimal policy is the MRAP to the turnpike, characterized in property 6.*

While the construction of $\bar{\mathbf{K}}(t)$ (cf. (2.12)) ensures $\sum_i K_i(0) = \sum_i \tilde{\tilde{K}}_i(0) = \bar{\mathbf{K}}(0)$, this condition does not imply the vector equality $K(0) = \tilde{\tilde{K}}(0)$. The elements of $K(0)$ are exogenously given (as outcomes of past investments), while the elements of $\tilde{\tilde{K}}(0)$ are determined optimally by the auxiliary problem. If past investment policies were suboptimal, the two capital vectors differ.

When $K(0) \neq \tilde{\tilde{K}}(0)$ we introduce a preliminary stage during which the capital stocks are rearranged to satisfy the desired condition. Specifically, the capital $\tilde{\tilde{K}}_i(0) - K_i(0)$ is added to $K_i(0)$, $i = n, s, r, d$. This rearrangement clearly equates the initial capital of each source to its initial auxiliary counterpart. Moreover, $\sum_i K_i(0) = \sum_i \tilde{\tilde{K}}(0)$ implies $\sum_i \left(\tilde{\tilde{K}}_i(0) - K_i(0)\right) = 0$, hence no additional investment is needed to reshuffle the stocks. Following this stage, the resulting initial capital stocks satisfy the condition of Property 6 and the auxiliary policy takes over.

The optimal policy can now be characterized as follows:

Proposition 2. *The optimal policy proceeds along the following stages:*

(*i*) *The initial capital stocks $K_i(0)$ are rearranged such that the capital stock of the i's source equals $\tilde{\tilde{K}}_i(0)$, $i = n, s, r, d$. This stage requires no additional capital.*

(*ii*) *Following the capital rearrangement, the optimal policy proceeds along the following stages, as specified in (3.25):*

(a) *the auxiliary policy is implemented until time τ, at which time all capital stocks have reached their turnpike counterparts. During $t \in [0, \tau]$, the optimal investments equal $\tilde{\tilde{x}}_i(t)$, specified in (3.24), and the entire investment budget $\bar{x}$ is utilized, hence the turnpikes are approached as rapidly as possible.*

(b) *from time τ onward, the turnpike policy corresponding to $\tilde{v}(Q^*(\tau))$ is implemented, with the associated turnpike investments specified in (3.12).*

(c) *The optimal processes eventually converge to the steady state specified in Proposition 1.*

4 Concluding comments

This work formulates water policy rules in the context of a prototypical water economy consisting of three water sources (natural, recycled and desalinated) and four user sectors (domestic, agriculture, industry and environment). A water policy consists of water allocation from each source to each user sector at each point of time and the capital investments needed to carry out these allocations. In spite of the complex structure of the water economy, the optimal policy rules are rather simple and straightforward, evolving along two main stages: a transition stage, during which the water capital stocks are brought as rapidly as possible to well-specified time-varying turnpikes (targets), and a turnpike stage, during which the water allocations and capital stocks evolve along their turnpike trajectories and eventually enter a steady state.

The analysis lands itself naturally to the pricing policy that implements the optimal water allocation. The ensuing water prices are source and sector specific, implying, for example, that the price of natural water allocated to households and industrial users differs from that allocated to agriculture and environment users. Recycled water is derived from treated domestic and industrial sewage and the latter is assumed mandatory, disregarding whether or not the treated sewage is reused later on. The implications are that domestic and industrial users should pay for sewage collection and treatment while users of recycled water should pay only for the cost of conveyance from the treatment plants to potential users as well as for extra treatment costs demanded by these users (but not by environmental regulation).

In general, a water price consists of three components: marginal cost, capital cost and scarcity cost – all expressed in dollar per cubic meter units. As these components vary across users and sources, so do the optimal water prices. The scarcity prices are associate with natural water and recycled water. The former is obvious in regions where natural water sources are insufficient to meet water demand. The latter accounts for the fact that recycled water is restricted by the flow of sewage, which in turn depends on water allocation to the residential and industrial sectors. As a result, the scarcity price of recycled water acts as a subsidy for users that generate sewage and as a tax for users that consume recycled water and could have far reaching implications regarding water allocation. For example, it encourages reallocation of natural water from irrigators to domestic users, as each reallocated cubic meter can generate about 0.65 cubic meter of recycled water that is solely due to the reallocation.

Desalination is an unlimited but expensive source. Its use, therefore, is justified only under severe water scarcity. Demographic and climatic trends imply that the number of such regions increases with time. The model presented herein can be used to determine the onset and extent of desalination over time.

The broad perspective taken in this work allows a sharp characterization of optimal policy rules, but inevitably leads to simplifications and abstractions. Extensions are needed to allow for arbitrary number of sources and user sectors, as well as to non-stationary economies with growing water demands and improved desalination technology (see discussion in Tsur 2016). A notable abstraction is the assumption of deterministic water supplies and demands. In actual practice, natural water supplies often fluctuate randomly with precipitation and the latter affects some (e.g., agricultural) water demands (see, e.g., Tsur 1990, Tsur and Graham-Tomasi 1991, Provencher and Burt 1994, Knapp and Olson 1995, Leizarowitz and Tsur 2012). These aspects can have profound effects on optimal policies and should, when relevant, be incorporated in actual applications.

The analytical approach of breaking a complicated problem into simpler sub-problems, applied here in the context of water resources management, extends well beyond this particular case. Indeed, this methodology can be used to simplify many investment problems, where complex inter-sectoral dependencies render direct analytical characterization of optimal policies intractable.

Appendix

Proof of Property 4. Noting (3.20) and letting $\bar{\lambda}(t)$ represent the costate of $\bar{\mathbf{K}}(t)$, The current-value Hamiltonian of the auxiliary problem is

$$\mathcal{H}(t) = \sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=r,s,d} C_i(q_{i\circ}(t)) - (\rho + \delta) \sum_i K_i(t) \\ + \theta(t)(R(Q(t)) - q_{n\circ}(t)) + \bar{\lambda}(t)(\bar{x} - \delta \bar{\mathbf{K}}(t)),$$

and the Lagrangian is

$$\mathcal{L}(t) = \mathcal{H}(t) + \sum_i \mu_i(t)[\gamma_i K_i(t) - q_{i\circ}(t)] + \xi(t)[\beta(q_{\circ D}(t) + q_{\circ I}(t)) - q_{r\circ}(t)] \\ + \vartheta(t) Q(t) + \eta(t) \left(\bar{\mathbf{K}}(t) - \sum_i K_i(t) \right),$$

where $\eta(t) \geq 0$ is the Lagrange multiplier of (2.13).

The necessary conditions for an optimum (with $K_i > 0$) include (nDI)-(dAE), (3.6) and (3.7), with the double-tilde replacing the single-tilde,

$$-(\rho + \delta) - \tilde{\tilde{\eta}}(t) + \tilde{\tilde{\mu}}_i(t)\gamma_i = 0, \tag{A.1}$$

$$\dot{\bar{\lambda}}(t) - \rho \bar{\lambda}(t) = \delta \bar{\lambda}(t) - \tilde{\tilde{\eta}}(t), \tag{A.2}$$

and the complimentary slackness condition

$$\tilde{\tilde{\eta}}(t) \left(\bar{\mathbf{K}}(t) - \sum_i \tilde{\tilde{K}}_i(t) \right) = 0. \tag{A.3}$$

Condition (A.1) can be rendered as

$$\tilde{\tilde{\eta}}(t) = \gamma_i \left(\tilde{\tilde{\mu}}_i(t) - \tilde{\mu}_i \right). \tag{A.4}$$

where $\tilde{\mu}_i$, defined in (3.2), is the shadow price of the capacity constraints $q_{i\circ} \leq \gamma_i K_i$ under the turnpike problem. Noting that $\tilde{\tilde{\eta}} \geq 0$, we find that $\tilde{\tilde{\mu}}_i(t) \geq \tilde{\mu}_i > 0$ (cf. (3.2)), hence

$$\tilde{\tilde{K}}_i(t) = \tilde{\tilde{q}}_{i\circ}(t)/\gamma_i, \tag{A.5}$$

verifying (3.22). Note again that (A.5) holds also when $\tilde{\tilde{K}}_i(t) = 0$ (even though (A.1) and (A.4) may not be valid in this case).

Noting property 2, the turnpike policy is optimal for the auxiliary problem if it is feasible. For $t < \tau$, the turnpike processes violate (2.13), inflicting some loss of value. Thus, the Lagrange multiplier $\tilde{\tilde{\eta}}(t)$ associated with the capital constraint (2.13) must obtain a positive value while $t < \tau$. The slackness condition (A.3), then, verifies (3.23). The differentiability of the $\tilde{\tilde{K}}_i(t)$'s, which gives rise to (3.24), is verified in Lemma 1 below. □

Lemma 1. *Under assumption 1, the $\tilde{\tilde{q}}_{ij}(t)$'s are continuously differentiable.*

Proof. Consider the following modified auxiliary problem:

$$\max_{q(t)\geq 0} \int_0^\tau \left(\sum_j B_j(q_{\circ j}(t)) - C_n(Q(t), q_{n\circ}(t)) - \sum_{i=s,r,d} C_i(q_{i\circ}(t)) - \sum_i \tilde{\tilde{\mu}}_i(t) q_{i\circ}(t) \right) e^{-\rho t} dt$$

subject to

$$\dot{Q}(t) = R(Q(t)) - q_{n\circ}(t),$$

$Q(0)$ given and the feasibility constraints of the auxiliary problem except (2.13), where the $\tilde{\tilde{\mu}}_i(t)$'s are the Lagrange multipliers of the $q_{i\circ}(t) \leq \gamma_i K_i(t)$ constraints under the auxiliary policy (see (A.1) above). It is straightforward to verify that the auxiliary processes $\tilde{\tilde{Q}}(t)$ and $\tilde{\tilde{q}}(t)$ are optimal for the modified auxiliary problem (in particular, they satisfy the necessary conditions of the modified auxiliary problem) and the associated capital stocks, the $\tilde{\tilde{K}}_i(t)$'s defined in (A.5), satisfy (2.13).

Let $L(Q(t), q(t), t)$ be the integrand of the above objective and $f(Q(t), q(t)) = R(Q(t)) - q_{n\circ}(t)$. Assumption 1 ensures that $L(\tilde{\tilde{Q}}(t), q)$ is strictly concave in q. Moreover, $f(\tilde{\tilde{Q}}(t), q)$ is linear in q. Thus, the conditions of Corollary 6.1 of Fleming and Rishel (1975, p. 77) are satisfied, implying that the $\tilde{\tilde{q}}_{ij}(t)$'s are continuously differentiable. It follows that $\tilde{\tilde{K}}_i(t) = \tilde{\tilde{q}}_{i\circ}(t)/\gamma_i$, $i = n, s, r, d$, are continuously differentiable as well, verifying that the $\tilde{\tilde{x}}_i(t)$'s, defined in (3.24), exist. □

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