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Steady-state properties in a class of dynamic models, with applications to natural resource management

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Abstract

We develop a method to characterize the location as well as the time of approach of optimal steady states in single-state, infinite-horizon, autonomous models. The method is based on a simple function of the state variable which is defined in terms of the model's primitives. The actual implementation does not require to solve the underlying dynamic optimization problem (which often does not admit a closed-form solution). Applying the method to a generic class of resource management problems, we show how it identifies the set of candidate steady states and determines, for each steady state, whether the corresponding approach time is finite or infinite.

Keywords: infinite horizon, autonomous problems, optimal policy, steady-state, approach time, natural resources, depletion time.

JEL classification: C61, Q20, Q30

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1 Introduction

An important property of intertemporal economic systems concerns their long-run stability or, more precisely, whether or not they converge to a steady state. When they do, two questions immediately arise regarding the location of the optimal steady state and the time it takes to approach it. While the conditions needed for the stability of multi-state processes are often hard to specify and/or verify (see Sorger 1989, and references cited there), the conditions ensuring the global stability of single-state, autonomous processes are rather straightforward. In this work we study the optimal steady-state location and approach time of the latter type of processes. For this purpose, we introduce a simple method based on a function of the state variable which can be obtained from the model's primitives without the need to actually solve the underlying dynamic optimization problem. We then apply the method to a generic class of resource management models.

In the context of natural resource management, the steady-state location concerns issues such as whether the resource should eventually be depleted (led to extinction) or some finite stock must always be reserved for future use. Alternatively, one can investigate the conditions or regulatory measures that give rise to more conservative exploitation relative to some benchmark, by comparing the location of the corresponding steady states (for examples of this type of analysis, see Tsur and Zemel 2004). The steady-state approach time has initially been examined in the context of nonrenewable resources (e.g., minerals) that are both finite and essential (see Dasgupta and Heal 1974, and references cited therein), with the main insight that the larger the alternative price of the mineral resource, the longer it should take to deplete it (in the limit, when the alternative price is infinite, depletion occurs asymptotically). In other settings, questions regarding the steady-state approach time are sometimes posed in such concrete terms as "when is it optimal to exhaust a resource in finite time?" (Akao and Farzin 2007) or "when should we stop extracting a nonrenewable resource?" (Schumacher 2011). Similar questions arise in the context of capital accumulation (investment) and growth models, where the capital stock or capital per (quality-adjusted) labor converge in the long run to constant levels.

The steady-state location problem has been studied by Tsur and Zemel (2001) by means of a function of the state variable, denoted $L(\cdot)$ and defined in terms of the model's primitives, which serves to formulate necessary conditions for optimal steady states. This L-method identifies the set of candidate optimal steady states. When the method suggests a unique candidate, the optimal state trajectory converges to it from any initial state. Otherwise, the optimal steady state to which the system converges may depend on the initial state.

The present work extends Tsur and Zemel's (2001) analysis in two ways. First, an additional necessary condition is specified in terms of the slope of $L(\cdot)$, which significantly narrows down the list of candidates for optimal steady-state in models admitting multiple candidates. Second, the L function formalism is used to determine whether the approach time to the steady state is finite or infinite, thereby addressing the second question in a straightforward manner. It turns out that both questions (the steady state location and time of approach) require merely an algebraic study of the associated $L(\cdot)$ function which can be easily carried out without the need to solve the underlying dynamic optimization problem, which often admits no explicit (closed-form) solution.

The next section lays out the setup for a general infinite-horizon dynamic optimization problem, defines the corresponding $L(\cdot)$ function and briefly summarizes the results of Tsur and Zemel (2001). Section 3 presents the new results – the extended necessary condition for an optimal steady state as well as the characterization of the steady-state approach time. Section 4 applies our method to a generic model of natural resource management, considering, in turn, nonrenewable and renewable resources. The formal proofs are presented in Section 5 and Section 6 concludes.

2 Setup

Let $X(t)$ and $c(t)$ denote, respectively, the state (stock) and control (action flow) of an economic system at time t. The action $c(t)$ affects the state's evolution according to

$$\dot{X}(t) = g(X(t), c(t)) \tag{2.1}$$

and gives rise to the instantaneous benefit $f(X(t), c(t))$. The policy $\{c(t);\ t \geq 0\}$ generates the payoff

$$\int_0^\infty f(X(t), c(t))e^{-\rho t} dt, \tag{2.2}$$

where ρ is the time rate of discount. The instantaneous benefit and the state dynamics functions, $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$, are twice continuously differentiable and satisfy[1]

$$f_c > 0, \quad f_{cc} < 0, \quad g_c \leq -\alpha < 0, \quad g_{cc} \leq 0, \quad f_X \geq 0, \tag{2.3}$$

for all $X \in (\underline{X}, \bar{X})$ and feasible c, where α is a given positive constant and the subscripts X and c denote partial derivatives with respect to these variables.

A feasible policy satisfies $X(t) \in [\underline{X}, \bar{X}]$ and $c(t) \in \mathcal{C} \equiv [\underline{c}, \bar{c}]$ for all $t \geq 0$, where $\underline{X} < \bar{X}$ are given state bounds and $\underline{c} < \bar{c}$ are given action bounds. The lower and upper bounds $\underline{X}$ and $\bar{X}$ can be determined by physical or regulatory constraints (e.g., natural resource stocks and produced capital stocks cannot turn negative, or a regulator may impose some positive lower bound on these stocks). Alternatively, these bounds can be induced by the action feasibility constraint $c \in \mathcal{C}$ (e.g., the initial stock of a nonrenewable resource or the carrying capacity stock of a renewable biomass resource determine the upper bound $\bar{X}$ if the exploitation rate is restricted to be non-negative). This distinction will turn out to be important in the analysis below.

The optimal policy is the feasible policy that maximizes (2.2) subject to (2.1) given $X(0) = X_0$. We assume that an optimal policy exists and denote the corresponding value function (the payoff under the optimal policy) by $v(X_0)$. Since the (single state) dynamic optimization problem under consideration is infinite-horizon and autonomous, the ensuing optimal state trajectory is monotonic in time (see Hartl 1987). As it is also bounded, the optimal state process must converge to a steady state, which we denote by $\hat{X} \in [\underline{X}, \bar{X}]$.

[1]The assumptions regarding the signs of f_c and g_c might appear restrictive (e.g., in investment models, where $X(t)$ is the capital stock and $c(t)$ is the investment rate, $g_c > 0$ and $f_c < 0$) but in fact they imply no loss of generality: one can always formulate the model in terms of the control $-c$ instead of c, keeping the requirement that $f_c g_c < 0$. In either case, the condition $|g_c| \geq \alpha > 0$ implies that c is influential in controlling the state evolution. A similar comment applies to f_X when X represents a damaging state, e.g., a pollution stock. We maintain the sign convention of Assumption 2.3 for the sake of concreteness.

Suppose that the *constant-state* function $M(X)$, defined by

$$g(X, M(X)) = 0, \tag{2.4}$$

is single-valued and corresponds to a feasible policy for all $X \in [\underline{X}, \bar{X}]$. It follows from (2.3)-(2.4) that

$$M'(X) = -g_X(X, M(X))/g_c(X, M(X)) \tag{2.5}$$

is well defined. Adopting the policy $c = M(X)$ leaves the process at the state X indefinitely, yielding the payoff

$$W(X) \equiv f(X, M(X))/\rho \leq v(X), \tag{2.6}$$

where the rightmost relation holds as an equality only at the optimal steady state $\hat{X}$. Define the function

$$L(X) \equiv \rho f_c(X, M(X))/g_c(X, M(X)) + \rho W'(X),$$

which, noting (2.5), can be expressed as

$$L(X) = \frac{f_c(X, M(X))}{g_c(X, M(X))}[\rho - g_X(X, M(X))] + f_X(X, M(X)). \tag{2.7}$$

The function $L(\cdot)$ serves to formulate the following necessary conditions for the location of the optimal steady state $\hat{X}$ (see Tsur and Zemel 2001):

Proposition 1 (necessary conditions for the location of $\hat{X}$)**.** $\hat{X} \in (\underline{X}, \bar{X})$ *only if* $L(\hat{X}) = 0$*;* $\hat{X} = \underline{X}$ *only if* $L(\underline{X}) \leq 0$*;* $\hat{X} = \bar{X}$ *only if* $L(\bar{X}) \geq 0$*.*

The proofs of all propositions are presented in Section 5.

We refer to the states where $L(\cdot)$ vanishes as the *roots* of L. Proposition 1 identifies the optimal steady state as either a root of L or one of the state bounds. It determines $\hat{X}$ uniquely in the following cases:

Corollary 1. *(i) If $L(\cdot)$ crosses zero once from above in $[\underline{X}\,,\bar{X}]$, then $\hat{X}$ falls on the unique root of $L(\cdot)$. (ii) If $L(X) > 0$ for all $X \in [\underline{X}\,,\bar{X}]$, then $\hat{X} = \bar{X}$. (iii) If $L(X) < 0$ for all $X \in [\underline{X}\,,\bar{X}]$, then $\hat{X} = \underline{X}$.*

In other cases, e.g., when $L(\cdot)$ obtains multiple roots in $[\underline{X}\,,\bar{X}]$, the proposition cannot determine $\hat{X}$ uniquely, although it restricts significantly the list of candidate steady states. In the next section we extend the proposition by restricting this list even further and use $L(\cdot)$ to determine whether the steady-state is approached at a finite time or asymptotically.

3 Steady states properties

We begin with the following useful extension of Proposition 1:

Proposition 2. *A root of $L(\cdot)$ can be an optimal steady state only if $L'(X) < 0$.*

Proposition 2 narrows the list of candidates for an optimal steady-state by ruling out roots of $L(\cdot)$ in which $L(\cdot)$ crosses zero from below. Processes initiated in the vicinity of such roots will be either repelled to the other direction or else proceed to this root and continue past it on the way towards a steady state behind it.

We turn now to study whether the steady-state approach time is finite or infinite. To that end, we distinguish between steady states at which $L(\cdot)$ vanishes and those at which $L(\cdot)$ does not. We refer to the former as *unconstrained* steady states and to the latter as *constrained* steady states. Note that unconstrained steady states (with $L(\hat{X}) = 0$ and $L'(\hat{X}) < 0$) can fall anywhere in $[\underline{X}\,,\bar{X}]$, including the upper and lower bounds, whereas constrained steady-states must fall on one of the bounds ($\underline{X}$ or $\bar{X}$). The modifier "unconstrained" indicates that the steady-state $\hat{X}$ would remain optimal even if the constraint $X(t) \in [\underline{X}, \bar{X}]$ were slightly relaxed, whereas the "constrained" modifier indicates that a change in the relevant bound ($\underline{X}$ or $\bar{X}$) would entail a different steady state. Let T denote the time it takes the optimal state process to approach the steady-state. The following result characterizes T for unconstrained steady states:

Proposition 3 (approach time to unconstrained steady states)**.** *Under assumption (2.3) and $X(0) \neq \hat{X}$, the approach to unconstrained steady states is asymptotic, i.e., $T = \infty$.*

Roughly speaking, this property stems from the continuity of the optimal action (control) process. As the state process approaches the steady state, the control approaches the constant-state rate $M(\hat{X})$, so the rate of further change in the state becomes small. Note the importance of the curvature condition regarding f in assumption (2.3) for this characterization. Indeed, if both f and g are linear in the control, a most rapid approach path (Spence and Starrett 1975) can bring the process to $\hat{X}$ within a minimal (finite) time T, followed by a discontinuous jump of $c(t)$ to the constant-state control $M(\hat{X})$ at $t = T$.

Constrained steady states (where $L(\hat{X}) \neq 0$) must fall on one of the bounds $\underline{X}$ or $\bar{X}$ (see Proposition 1), and the approach time to such steady states depends on the following two classifications: (1) Is the feasibility constraint $c \in \mathcal{C}$ binding at the steady state? (it is *not* binding if $M(\hat{X})$ lies in the interior of $\mathcal{C}$); and (2) is the resource *essential* at $\hat{X}$? i.e., whether or not $f_c(X, M(X)) \to \infty$ as $X \to \hat{X}$. Typically, the latter property is relevant only at the lower bound, where $\hat{X} = \underline{X}$ and $M(\underline{X})$ is too small to meet essential needs (e.g., non-renewable resources that are being depleted or regenerating resources on the way towards extinction). Indeed, if $\hat{X} = \bar{X}$, the resource is not essential at $\hat{X}$. To see this, recall that $f_{cc} < 0$ hence the divergence of f_c implies that $M'(X) < 0$ in the vicinity of $\bar{X}$. Thus, $g_X(\bar{X}, M(\bar{X})) < 0$ (see (2.5)) hence $\rho - g_X(\bar{X}, M(\bar{X})) > 0$ and (2.7) implies $L(\bar{X}) = -\infty$, violating the condition in Proposition 1 for $\bar{X}$ to be an optimal steady state. At the other bound, where $\hat{X} = \underline{X}$, we refer to a resource as *essential* or *nonessential* depending on whether $f_c(\underline{X}, M(\underline{X}))$ is infinite or finite, respectively.

The properties of T for constrained steady states are summarized in:

Proposition 4 (approach time to constrained steady-states). *Suppose* $\hat{X} \neq X_0$ *and* $L(\hat{X}) \neq 0$ *(so* $\hat{X}$ *must fall on either the upper or lower state bound):* (i) *If the feasibility constraint* $c \in \mathcal{C}$ *is not binding at* $\hat{X}$*, then* $T = \infty$ *or* $T < \infty$ *depending on whether the resource is essential or nonessential, respectively.* (ii) *If the feasibility constraint* $c \in \mathcal{C}$ *is binding at* $\hat{X}$ *and* $|g_X(\hat{X}, M(\hat{X}))| < \infty$*, then* $T = \infty$*.*

The asymptotic steady-state approach ($T = \infty$) in case (ii) owes to the feasibility constraint on the action c near the steady state. In case (i) this constraint is not binding, but for essential resources the divergence of the marginal benefit near the steady state acts as an effective c-constraint, giving rise again to asymptotic steady-state approach.

In summary, optimal, bounded, one-dimensional state trajectories of infinite-horizon, autonomous problems converge to a steady state. Propositions 1 and 2 identify the set of candidates for an optimal steady-state and Propositions 3 and 4 determine whether the steady-state approach time is finite or infinite. These properties are formulated in terms of the function $L(\cdot)$ which can be obtained from the model's primitives without the need to actually solve the underlying dynamic optimization problem. The latter feature is particularly useful in the context of problems which do not admit a closed-form solution and the determination of T using standard methods is not trivial, as demonstrated in the next section.

4 Application to resource management

We apply the above framework to natural resource management models, where $X(t)$ represents the remaining resource stock and $c(t)$ is the rate of exploitation (mining, extraction, harvesting) at time t. The net benefit function takes the form $f(X, c) = u(c) - Z(X)c$, where $u(\cdot)$ is an increasing and strictly concave utility function and $Z(\cdot) \geq 0$ is a non-increasing and convex unit extraction cost. The resource dynamics is specified as

$$g(X, c) = R(X) - c \tag{4.1}$$

where $R(\cdot)$ is the recharge (growth, regeneration) function. For nonrenewable resources, $R(X) = 0$ for all X, whereas for renewable resources $R(X)$ is positive over some interval $(0, \bar{X})$. The resource exploitation policy $\{c(t),\, t \geq 0\}$ is feasible if $c(t) \geq 0$ and $X(t) \geq 0$ for all $t \geq 0$. This policy generates the payoff

$$\int_0^\infty [u(c(t)) - Z(X(t))c(t)]e^{-\rho t}dt \tag{4.2}$$

and the optimal policy is the feasible policy that maximizes (4.2) subject to the state dynamics constraint (4.1) given the initial stock X_0.

This formulation is widely used in the resource economics literature and many properties of the ensuing optimal policies are well known (see, e.g., Clark 1976, Dasgupta and Heal 1979). It therefore serves well the purpose of demonstrating the use of the above analysis for locating the optimal steady states and determining the associated approach times without actually solving the dynamic optimization problems. We discuss nonrenewable and renewable resources in turn.

4.1 Nonrenewable resources

With a vanishing $R(X)$, (2.1) specializes to

$$\dot{X}(t) = -c(t). \tag{4.3}$$

The state process is non-increasing hence the upper bound is set as $\bar{X} = X_0$. The constant-state function $M(X)$ vanishes identically for all X and $L(\cdot)$ of (2.7) specializes to

$$L(X) = \rho[Z(X) - u'(0)]. \tag{4.4}$$

Consider first a non-essential resource with $u'(0) < \infty$. The function $L(\cdot)$ decreases and Proposition 1 identifies the unique optimal steady-state

$$\hat{X} = \begin{cases} 0 & \text{if } Z(0) - u'(0) \leq 0 \\ Z^{-1}(u'(0)) & \text{if } Z(0) - u'(0) > 0 \text{ and } Z(X_0) - u'(0) < 0 \\ X_0 & \text{if } Z(X_0) - u'(0) \geq 0 \end{cases}, \tag{4.5}$$

thereby determining whether the resource will be depleted ($\hat{X} = 0$), exploited but not depleted ($\hat{X} \in (0, X_0)$) or not exploited at all ($\hat{X} = X_0$).

From (4.4)-(4.5), we see that if $u'(0) \in (Z(X_0), Z(0)]$, then $\hat{X} = Z^{-1}(u'(0))$ is unconstrained ($L(\hat{X}) = 0$). Thus, the approach to $\hat{X}$ is asymptotic, with $T = \infty$ (cf. Proposition 3). If $u'(0) > Z(0)$ then $\hat{X} = 0$ is constrained ($L(\hat{X}) < 0$) and according to Proposition 4(i), depletion occurs at a finite time. In this case, the marginal benefit $u'(0)$ is sufficiently large to justify early depletion, but the resource is nonessential (since $u'(0) < \infty$). If $u'(0) \leq Z(X_0)$, the extraction cost exceeds the marginal benefit at all states and the resource does not admit profitable exploitation ($c = 0$, $\hat{X} = X_0$ and $T = 0$).

When $u'(c) \to \infty$ as $c \downarrow 0$, $L(X) = -\infty$ at all $X \in [0, X_0]$ and Proposition 1 implies depletion ($\hat{X} = 0$). Since $f_c(\hat{X}, M(\hat{X})) = u'(0) - Z(0) = \infty$, the resource is essential and Proposition 4(i) implies asymptotic depletion ($T = \infty$).

The special case in which the extraction cost is constant, say $Z(X) = z$, corresponds to Hotelling's (1931) model. In this case $L(X) = \rho[z - u'(0)]$ is constant and Proposition 1 implies that $\hat{X} = X_0$ or $\hat{X} = 0$ depending on whether $z \geq u'(0)$ or $z < u'(0)$, respectively. In the former case the resource does not admit profitable exploitation. In the latter case the steady state $\hat{X} = 0$ is constrained (since $L(0) < 0$) and according to Proposition 4, the steady state will be approached at a finite time or asymptotically depending on whether the resource is non-essential ($u'(0)$ is finite) or essential ($u'(0) = \infty$).

4.2 Renewable resources

The stock of a renewable resource evolves according to

$$\dot{X}(t) = R(X(t)) - c(t), \tag{4.6}$$

where the recharge function $R(\cdot)$ varies across resource types. In all cases we assume the existence of some state $\bar{X} > 0$ (corresponding to the resource maximal volume or carrying capacity) with $R(\bar{X}) = 0$ and $R(X) \leq 0$ for all $X > \bar{X}$. With exogenous recharge, e.g. precipitation feeding water sources, $R(\cdot)$ is typically decreasing and concave over $[0, \bar{X}]$. When the recharge is due to growth (regeneration) as in biomass resources (e.g. a fishery or forests) $R(\cdot)$ initially increases and reaches a peak at the maximum-sustainable-yield stock, X_{MSY}, and decreases thereafter to vanish at the carrying capacity stock $\bar{X}$.

The management problem entails finding the feasible policy $\{c(t),\, t \geq 0\}$ that maximizes (4.2) subject to (4.6), given the initial stock X_0. The constant-stock function is $M(X) = R(X)$ and equation (2.7) specializes to

$$L(X) = -[\rho - R'(X)][u'(R(X)) - Z(X)] - Z'(X)R(X)\,. \tag{4.7}$$

We discuss water and biomass resources in turn.

4.2.1 Water resources

Suppose that the recharge function is decreasing and concave, as is typically the case for water resources (see Tsur and Zemel 2004, and references cited therein). The state $\bar{X}$ represents a full reservoir, with $R(\bar{X}) = 0$ and $R'(\bar{X}) > -\infty$. Differentiating (4.7) gives

$$\begin{aligned} L'(X) &= R''(X)[u'(R(X)) - Z(X)] \\ &- \{[\rho - R'(X)][u''(R(X))R'(X) - Z'(X)] + Z''(X)R(X) + Z'(X)R'(X)\}\,. \end{aligned} \tag{4.8}$$

The expression inside the curly brackets is positive. Setting $L(\hat{X}) = 0$ in (4.7) gives

$$u'(R(\hat{X})) - Z(\hat{X}) = \frac{-Z'(\hat{X})R(\hat{X})}{\rho - R'(\hat{X})} \geq 0\,, \tag{4.9}$$

hence the first term of (4.8) is negative at $X = \hat{X}$, ensuring that $L'(\hat{X}) < 0$ at any root $\hat{X}$. It follows that $L(\cdot)$ can have at most one root in $[0, \bar{X}]$, in which case the optimal steady state is uniquely identified by Proposition 1 as follows:

$$\hat{X} = \begin{cases} 0 & \text{if } L(0) < 0 \\ L^{-1}(0) & \text{if } L(0) \geq 0 \text{ and } L(\bar{X}) \leq 0 \\ \bar{X} & \text{if } L(\bar{X}) > 0 \end{cases}.$$

Denoting

$$\Lambda(X) \equiv \frac{-Z'(X)R(X)}{\rho - R'(X)} \geq 0\,, \tag{4.10}$$

we specify the steady state in terms of the model's primitives

$$\hat{X} = \begin{cases} 0 & \text{if } \; u'(R(0)) > Z(0) + \Lambda(0) \\ L^{-1}(0) & \text{if } u'(R(0)) \leq Z(0) + \Lambda(0) \text{ and } u'(0) \geq Z(\bar{X}) \\ \bar{X} & \text{if } u'(0) < Z(\bar{X}) \end{cases} . \tag{4.11}$$

When $u'(R(0)) \leq Z(0) + \Lambda(0)$ and $u'(0) > Z(\bar{X})$, the steady-state is unconstrained and, according to Proposition 3, is approached asymptotically, with $T = \infty$. When $u'(R(0)) > Z(0) + \Lambda(0)$, the steady-state $\hat{X} = 0$ is constrained ($L(0) < 0$) and the resource is nonessential (since $R(0) > 0$ and $Z(0) > 0$ ensure that $u'(R(0)) - Z(0) < \infty$). Thus, according to Proposition 4(i), depletion occurs at a finite time. Finally, when $u'(0) \leq Z(\bar{X})$, the unit extraction cost exceeds the highest price the resource can obtain and the resource does not admit profitable exploitation. The constant-state rate $M(\bar{X}) = 0$ lies at the boundary of $\mathcal{C}$ hence, although the resource is nonessential at $\bar{X}$, the feasibility constraint $c \geq 0$ implies that the water reservoir is filled to its full capacity $\hat{X} = \bar{X}$ asymptotically (unless $X(0) = \bar{X}$, see Proposition 4(ii)).

The vanishing of $L(\cdot)$ at unconstrained steady states bears a simple economic interpretation. Writing (4.9) as

$$u'(R(\hat{X})) = Z(\hat{X}) + \Lambda(\hat{X}), \tag{4.12}$$

we recall that optimal management requires that at each point of time the marginal value of extraction equals the full cost of the resource, which consists of the unit extraction cost $Z(\cdot)$ plus the shadow price (scarcity rent, royalty, *in situ* value) of the resource. The second term on the right-hand side of (4.12) is identified by (4.10) as the shadow price of the resource at the steady state:

Increasing the stock by dX decreases the unit extraction cost by $-Z'(\hat{X})dX$ and the total extraction cost by $-Z'(\hat{X})R(\hat{X})dX$. The present value of this change in cost flow is imputed at the effective discount rate $\rho - R'(\hat{X})$ (which accounts also for the change in recharge rate $R(\cdot)$ due to the change in stock).

4.2.2 Biomass resources

For biomass resources, the function $R(\cdot)$ represents the natural growth rate which increases from $R(0) = 0$ at all $X \in [0, X_{MSY})$, reaches a peak at the maximum-sustainable-yield state X_{MSY} and decreases over $X \in (X_{MSY}, \bar{X}]$ until reaching zero again at the carrying capacity state $\bar{X}$. Over its decreasing domain, the function $R(\cdot)$ is concave.

If $L(\bar{X}) \geq 0$ then $u'(0) < Z(X)$ for all $X \in (0, \bar{X})$ and the resource does not admit profitable exploitation. To study harvesting policies, we suppose that $L(\bar{X}) < 0$. If $L(X) < 0$ for all $X > 0$ then, according to Proposition 1, the unique steady state is $\hat{X} = 0$, where the biomass resource is brought to extinction. If, in addition, $L(0) = 0$ then, according to Proposition 3, extinction occurs asymptotically. If $L(0) < 0$ then, according to Proposition 4-(i), extinction occurs asymptotically or at a finite time depending on whether the resource is essential (i.e., $u'(0)$ is unbounded) or nonessential ($u'(0)$ is finite).

When $L(0) > 0$ the function $L(\cdot)$ obtains at least one root in $(0, \bar{X})$, at which (4.12) holds. According to Propositions 1-2, only roots of $L(\cdot)$ at which $L'(\cdot) < 0$ are legitimate candidates for optimal steady states. The approach to such (unconstrained) states is asymptotic (unless $X(0) = \hat{X}$). If only one such root exists, it is the optimal steady state to which the optimal stock process converges from any $X(0) \in (0, \bar{X}]$. If multiple roots exist, depending on the specifications of $u(\cdot)$, $Z(\cdot)$ and $R(\cdot)$, each (legitimate) root may or may not have a nonempty basin of attraction. In some cases, a global maximum exists, which attracts the process from any initial state. In other cases, the optimal steady state varies with the initial stock. In such cases, the optimal root must be determined by evaluating the objective associated with each of the legitimate steady states, since the local analysis embodied in the $L-$function formalism cannot provide global information (see the discussion in Tsur and Zemel 2001).

5 Proofs

This section presents the proofs of Propositions 1-4. It also illustrates the optimal constrained steady-state arrival times, characterized in Proposition 4, by means of simple examples of the resource management problem of Section 4 for which explicit solutions are readily available. These examples illuminate the underlying factors determining whether the steady-state approach time is finite or infinite.

For the sake of completeness, we reproduce here the variational argument used by Tsur and Zemel (2001) to prove Proposition 1:

Proof of Proposition 1. For any feasible state X we compare the constant-state value $W(X)$ obtained from the policy $C = M(X)$ with the value obtained from a small variation from this policy. If the variation policy yields a value that exceeds $W(X)$, then the constant-state policy is not optimal at X and this state does not qualify as an optimal steady state. Choose the arbitrarily small constants $h > 0$ and δ and consider the following variation policy:

$$c^{h\delta}(t) = \begin{cases} M(X) + \delta/g_c(X, M(X)) & \text{if } t \leq h \\ M(X(h)) & \text{if } t > h \end{cases}.$$

For the short period $t \leq h$, this policy deviates slightly from the constant state policy, then it enters a steady state at $X(h)$. During the first period, $\dot{X} = \delta + o(h\delta)$ which brings the state at $t = h$ to $X(h) = X + h\delta + o(h\delta)$. The contribution of this period to the objective is evaluated (up to $o(h\delta)$) as[2]

$$\int_0^h f(X(t), M(X)+\frac{\delta}{g_c}) \exp(-\rho t) dt = \int_0^h \rho W(X) \exp(-\rho t) dt + \frac{f_c(X, M(X))}{g_c(X, M(X))} h\delta.$$

The contribution of the constant-state policy $c = M(X(h))$ during the period $t > h$ is approximated up to $o(h\delta)$ terms as

$$\begin{aligned} \int_h^\infty f(X(h), M(X(h))) \exp(-\rho t) dt &= \int_h^\infty \rho W(X(h)) \exp(-\rho t) dt \\ = \int_h^\infty \rho W(X) \exp(-\rho t) dt &+ W'(X) h\delta. \end{aligned}$$

Summing the contributions of the two periods gives the value $V^{h\delta}(X)$ obtained with $c^{h\delta}$. Recalling (2.7), we find

$$V^{h\delta}(X) - W(X) = L(X) h\delta/\rho + o(h\delta).$$

The sign of δ can be freely chosen, while $h > 0$. Now, if $L(X) \neq 0$ we can set sign(δ) =sign($L(X)$) which gives $V^{h\delta}(X) > W(X)$ and X is not an optimal steady state. Thus, only the roots of $L(\cdot)$ qualify as legitimate candidates for $\hat{X}$. The only possible exceptions are the bounds $\underline{X}$ and $\bar{X}$. Choosing $\delta > 0$ is not feasible at $\bar{X}$ because this policy would lead the process outside the feasible domain. It follows that $\bar{X}$ cannot be excluded as an optimal steady state if $L(\bar{X}) > 0$. A similar argument holds for the lower bound $\underline{X}$ if L is negative at this state. □

We introduce some general derivations and definitions that are used in the following proofs. A steady state $\hat{X}$ at which the constraint $c \in \mathcal{C} \equiv [\underline{c}, \bar{c}]$

[2] $o(h\delta)$ indicates terms such that $o(h\delta)/(h\delta) \to 0$ as $h\delta \to 0$.

is binding (i.e., $M(\hat{X}) \in \partial\mathcal{C} \equiv \{\underline{c}, \bar{c}\}$) is referred to as c-constrained. A steady state $\hat{X}$ is called s-constrained if it falls on one of the bounds, $\underline{X}$ or $\bar{X}$, which is a result of a physical or a regulatory constraint (see discussion below equation (2.3)). With $\lambda(t)$ denoting the current-value costate, the current-value Hamiltonian corresponding to the problem of maximizing the objective (2.2) subject to the dynamic constraint (2.1), given the initial state X_0, is

$$\mathcal{H} = f(X, c) + \lambda g(X, c).$$

The necessary conditions for (an interior) optimum include:

$$f_c(X, c) + \lambda g_c(X, c) = 0 \tag{5.1}$$

and

$$\dot{\lambda} - \rho\lambda = -[f_X(X, c) + \lambda g_X(X, c)]. \tag{5.2}$$

In infinite-horizon, autonomous problems, the optimal control can be expressed as a function of the state, say $c(t) = C(X(t))$. Since the Hamiltonian is strictly concave in c (Assumption (2.3)), $C(\cdot)$ is single-valued and continuous, hence

$$C(\hat{X}) = M(\hat{X}). \tag{5.3}$$

We consider first the case in which the steady state $\hat{X}$ is not c-constrained, i.e., $M(\hat{X})$ lies in the interior of $\mathcal{C}$ and, noting (5.3), $C(X) \notin \partial\mathcal{C}$ for all X in some vicinity of $\hat{X}$. In this vicinity, the feasibility constraints on c can be ignored and the interior optimum is obtained from the necessary conditions (5.1)-(5.2).

Define the functions

$$A(X) = g_c(X, C(X)) f_{cc}(X, C(X)) - f_c(X, C(X)) g_{cc}(X, C(X)), \tag{5.4}$$

$$B(X) = g_c(X, C(X)) f_{cX}(X, C(X)) - f_c(X, C(X)) g_{cX}(X, C(X)) \tag{5.5}$$

and

$$\psi(X, c) = -(\rho - g_X(X, c)) f_c(X, c)/g_c(X, c) - f_X(X, c). \tag{5.6}$$

Then, (5.1)-(5.2) imply

$$\dot{\lambda} = \psi(X, C(X)), \tag{5.7}$$

while (2.7) reduces to

$$L(X) = -\psi(X, M(X)). \tag{5.8}$$

Taking the time derivative of (5.1) and using (5.7) to eliminate $\dot{\lambda}$, we find

$$C'(X)\frac{A(X)}{g_c^2(X, C(X))} + \frac{B(X)}{g_c^2(X, C(X))} + \frac{\psi(X, C(X))}{g(X, C(X))} = 0. \tag{5.9}$$

Equation (5.9) is a first order differential equation, which together with (5.3) defines $C(X)$ for all X. A difficulty with its implementation at $\hat{X}$ arises because the function g, appearing at the denominator of the last term, vanishes at $\hat{X}$. We distinguish between unconstrained steady states, where $L(\hat{X}) = 0$, and constrained steady states, where $L(\hat{X}) \neq 0$.

5.1 Unconstrained steady states

Proof of Proposition 2. In an unconstrained steady state, $L(\hat{X}) = 0$ and the singularity of the last term of (5.9) at $\hat{X}$ is removed because $\psi(\hat{X}, C(\hat{X})) = \psi(\hat{X}, M(\hat{X})) = -L(\hat{X}) = 0$ (cf. (5.8)) This term, then, can be evaluated using l'Hôpital's rule. Using (5.8), we find

$$\frac{d\psi(\hat{X}, C(\hat{X}))}{dX} = -L'(\hat{X}) + \psi_c(\hat{X}, C(\hat{X}))[C'(\hat{X}) - M'(\hat{X})],$$

while (2.5) implies

$$\frac{dg(X, C(X))}{dX} = g_X(X, C(X)) + g_c(X, C(X))C'(X) = g_c(X, C(X))[C'(X) - M'(X)].$$

It follows that

$$\lim_{X\to\hat{X}}\left\{\frac{\psi(X,C(X))}{g(X,C(X))}\right\}=\frac{1}{g_c(\hat{X},C(\hat{X}))}\left(\frac{-L'(\hat{X})}{C'(\hat{X})-M'(\hat{X})}+\psi_c(\hat{X},C(\hat{X}))\right).$$

The last term on the right hand side is obtained by taking the derivative of (5.6) with respect to c,

$$\psi_c(X,C(X))=-A(X)\frac{\rho-g_X(X,C(X))}{g_c^2(X,C(X))}-\frac{B(X)}{g_c(X,C(X))},$$

which reduces (5.9) in the limit $X\to\hat{X}$ to

$$\frac{A(\hat{X})}{g_c(\hat{X},C(\hat{X}))}\left(C'(\hat{X})-M'(\hat{X})-\frac{\rho}{g_c(\hat{X},C(\hat{X}))}\right)+\frac{-L'(\hat{X})}{C'(\hat{X})-M'(\hat{X})}=0.$$

Denoting

$$\Delta(X)\equiv C'(X)-M'(X), \tag{5.10}$$

we obtain the quadratic equation

$$\Delta^2(\hat{X})-\frac{\rho}{g_c(\hat{X},C(\hat{X}))}\Delta(\hat{X})-\frac{g_c(\hat{X},C(\hat{X}))L'(\hat{X})}{A(\hat{X})}=0. \tag{5.11}$$

To determine which of the solutions of (5.11) corresponds to the optimal steady-state slope-difference $\Delta(\hat{X})$, observe that the state $\hat{X}$ is attractive only if $\Delta(\hat{X})>0$. To see this, consider a state just below the steady state, say $X_\varepsilon=\hat{X}-\varepsilon$. To approach $\hat{X}$ from below requires $\dot{X}=g(X_\varepsilon,C(X_\varepsilon))>0$. Recalling that $g(X_\varepsilon,M(X_\varepsilon))=0$ and $g_c<0$, this implies $C(X_\varepsilon)<M(X_\varepsilon)$, while $C(\hat{X})=M(\hat{X})$. Thus, $C'(\hat{X})>M'(\hat{X})$ and $\Delta(\hat{X})>0$.

Next, we write the solutions of (5.11) as

$$\Delta(\hat{X})=\frac{\rho}{-2g_c(\hat{X},C(\hat{X}))}\left(-1\pm\sqrt{1+\frac{4L'(\hat{X})g_c^3(\hat{X},C(\hat{X}))}{\rho^2A(\hat{X})}}\right). \tag{5.12}$$

Since $g_c<0$ and $A(\hat{X})>0$, the argument of the square-root operator above exceeds unity only if $L'(\hat{X})<0$. In this case, we have one positive solution for $\Delta(\hat{X})$ which can provide the boundary value $C'(\hat{X}) = M'(\hat{X}) + \Delta(\hat{X})$ for the differential equation (5.9). In contrast, if $L'(\hat{X}) > 0$, the argument falls short of unity and the two solutions of (5.12) are either negative or complex, hence (5.9) does not yield a solution that converges to $\hat{X}$. This rules out the possibility that $L'(\hat{X}) > 0$ at an optimal steady state, verifying Proposition 2.

□

Proof of Proposition 3. We now show that an unconstrained steady state $\hat{X}$, at which $L(\hat{X}) = 0$ and $L'(\hat{X}) < 0$, cannot be approached at a finite time, i.e., $T = \infty$ (except, of course, for the special case where $X(0) = \hat{X}$ which gives $T = 0$). Suppose to the contrary, that T is finite. Using the solution $C(\cdot)$ of (5.9), the optimal state trajectory $X(t)$ can be obtained implicitly for any $t \in [0, T]$ from the solution of (2.1):

$$T - t = \int_{X(t)}^{\hat{X}} \frac{dx}{g(x, C(x))} \ . \tag{5.13}$$

Assume, for the sake of concreteness, that $X(0) < \hat{X}$, so the state process increases toward $\hat{X}$, i.e., $\dot{X}(s) = g(X(s), C(X(s)) > 0$ during $s \in [t, T)$. Since $X(t) \to \hat{X}$, for every $\varepsilon > 0$ there exists some time t_ε such that $\hat{X} - X(t) < \varepsilon$ for all $t_\varepsilon \le t \le T$. Denote $X_\varepsilon = X(t_\varepsilon)$. Then, for all $X \in [X_\varepsilon, \hat{X}]$

$$\begin{aligned} g(X, C(X)) &= g(\hat{X}, C(\hat{X})) + [g_X(\hat{X}, C(\hat{X})) + g_c(\hat{X}, C(\hat{X}))C'(\hat{X}) + O(\varepsilon)](X - \hat{X}) \\ &= g(\hat{X}, M(\hat{X})) + [-g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X}) + O(\varepsilon)](\hat{X} - X) \\ &\le -2g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})(\hat{X} - X), \end{aligned}$$

where the last inequality follows when ε is chosen to be sufficiently small so that[3] $O(\varepsilon) < -g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})$. Thus

$$T - t_\varepsilon > \frac{1}{-2g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})} \int_{X_\varepsilon}^{\hat{X}} \frac{dx}{\hat{X} - x} \ . \tag{5.14}$$

[3] $O(\varepsilon)$ denotes terms such that $O(\varepsilon)/\varepsilon$ is bounded when $\varepsilon \to 0$.

The integral on the right side of (5.14) diverges for every $X_\varepsilon < \hat{X}$, contradicting the assumption that T is finite. The case of decreasing state processes, with $X(t) > \hat{X}$ and $g < 0$ is treated in a similar manner. $\square$

5.2 Constrained steady states

When the optimal steady state is constrained, i.e, $L(\hat{X}) \neq 0$ and the steady falls on one of the corners ($\underline{X}$ or $\bar{X}$), the above derivation of T cannot be applied because the term $\psi(\hat{X}, C(\hat{X}))/g(\hat{X}, C(\hat{X}))$ of (5.9) diverges since g vanishes while $L = -\psi$ does not. It follows that $C'(\hat{X})$, hence also $\Delta(\hat{X})$, diverges so the right hand side of (5.14) does not necessarily yield an infinite value. Indeed, if all the functions and derivatives in (2.3) are continuous and bounded at the unconstrained steady state, then T obtains a finite value. If, however, f_c diverges at the steady state, then T may diverge as well.

Proof of Proposition 4(i): essential resources. Recall that the resource can be essential at a constrained steady state only when the latter falls on the lower bound $\underline{X}$. Suppose that $\hat{X} = \underline{X}$, and $f_c(\underline{X}, M(\underline{X})) = \infty$. Since $L(\underline{X})$ must be negative at this steady state, we find from (2.7) that $L(\hat{X}) = -\infty$, so that $m \equiv \rho - g_X(\underline{X}, M(\underline{X})) > 0$. Write, recalling (5.1) and (5.6),

$$\frac{\psi}{\lambda} = \rho - g_X + \frac{f_X g_c}{f_c}$$

and observe that the first two terms on the right-hand side approach the constant m while the third term shrinks to 0 when $X(t) \to \underline{X}$. It follows that close enough to the steady state, $\dot{\lambda}/\lambda \approx m$ hence $\lambda(t) \approx \tilde{\lambda}\exp(mt)$, where $\tilde{\lambda}$ is some positive constant. Fast as this exponential growth may be, it cannot take the $\lambda(\cdot)$ process to its target value $\hat{\lambda} = -f_c(\underline{X}, M(\underline{X}))/g_c(\underline{X}, M(\underline{X})) = \infty$ within a finite period of time. We conclude that $T = \infty$ in this case. $\square$

As an example, consider the simplest nonrenewable resource management problem obtained under the specifications $f = c^\beta$ with $0 < \beta < 1$ and $g = -c$, subject to $X(t) \geq 0$, given $X(0) = X_0$. In this problem, $M(X) = 0$ for all X, so $f_c(X, M(X)) = \beta M(X)^{\beta-1} = \infty$ and the resource is essential. Moreover, $L(X) = -\rho\beta M(X)^{\beta-1} = -\infty$ for all X. Thus, the lower bound $\underline{X} = 0$ is the unique steady state. Let $\alpha \equiv \rho/(1-\beta) > 0$. It is easy to verify that the optimal processes for this problem are given by

$$c(t) = \alpha X_0 e^{-\alpha t}$$

$$X(t) = X_0 e^{-\alpha t}\,.$$

Although this solution converges to the corner state with $L(0) \neq 0$, we have $T = \infty$ in this case, due to the divergence of $f_c(0, 0)$. Indeed, this solution implies $C(X) = \alpha X$, which is consistent with the value $C'(X) = \alpha$ obtained from (5.9) with the specifications $A = \beta(1-\beta)C^{\beta-2}$, $B = 0$, $\psi = \rho\beta C^{\beta-1}$, $g = -C$ and $g_c = -1$. With $M'(0) = 0$, $\Delta(0)$ is finite and the argument based on (5.14) establishes the asymptotic approach to the s-constraint steady state.

Proof of Proposition 4(i): non-essential resources. Consider, first, the case in which the upper bound $\bar{X}$ is a steady state where $L(\bar{X})$ is positive but finite. We show that T is finite in this case. Suppose otherwise, that T is infinite. Since all the functions listed in assumption (2.3) are continuous and bounded as $X(t) \to \bar{X}$ monotonically, then for any $\varepsilon > 0$ there exists some time t_ε such that $|\psi(X(t), C(X(t))) - \psi(\bar{X}, C(\bar{X}))| \leq \varepsilon$ for all $t > t_\varepsilon$. Recalling (5.8) and $C(\bar{X}) = M(\bar{X})$, we find $\psi(X(t), C(X(t))) \leq \varepsilon - L(\bar{X})$. Choosing $\varepsilon = L(\bar{X})/2 > 0$, equation (5.7) implies $\dot{\lambda} \leq -L(\bar{X})/2$ for all $t > t_\varepsilon$. Such a constant decrease in $\lambda(\cdot)$ cannot continue indefinitely because it would bring the shadow price process below the finite steady state value $\hat{\lambda} = -f_c(\bar{X}, M(\bar{X}))/g_c(\bar{X}, M(\bar{X}))$ in a finite time period, so the end conditions would be violated. We conclude that T must be finite.

The treatment of the case in which the lower bound $\underline{X}$ is a steady state where $L(\underline{X})$ is negative but finite is similar. If $T = \infty$, one finds that $\dot{\lambda}$ is larger than the positive constant $-L(\underline{X})/2$ for all $t > t_\varepsilon$, implying that $\lambda(\cdot)$ exceeds the finite steady state value $\hat{\lambda} = -f_c(\underline{X}, M(\underline{X}))/g_c(\underline{X}, M(\underline{X})$ after a finite time, violating the end conditions. The steady-state entrance time T, then, must be finite also in this case. $\square$

As an explicit example for the characterization of a non-essential resource, consider again the nonrenewable resource with $g = -c$ but change the specification of the benefit function to $f = \beta c - c^2/2$, where $\beta > 0$ is a given constant. We find again $M(X) = 0$ hence $f_c(X, M(X)) = \beta < \infty$ and the resource is not essential. For any state X, $L(X) = -\rho\beta < 0$ hence only the lower bound $\underline{X} = 0$ can be an optimal steady state. The optimal policy in this case is given by

$$c(t) = \begin{cases} \beta - \lambda_0 e^{\rho t} & \text{if } \ t < T \\ 0 & \text{if } \ t \geq T \end{cases}, \tag{5.15}$$

where the constants λ_0 and T are determined by the conditions

$$\lambda_0 e^{\rho T} = \beta \ \ \leftrightarrow \ \ c(T) = 0; \qquad X_0 - \beta T + \lambda_0 \int_0^T e^{\rho t} dt = 0 \ \ \leftrightarrow \ \ X(T) = 0.$$

Eliminating λ_0, we determine T implicitly from the equation

$$e^{-\rho T} = 1 + \frac{\rho X_0}{\beta} - \rho T \tag{5.16}$$

which admits a unique finite solution $0 < T < X_0/\beta + 1/\rho$, while

$$\lambda_0 = \beta + \rho X_0 - \beta\rho T. \tag{5.17}$$

The $C(\cdot)$ function is expressed implicitly as

$$\rho[X(t) - X_0] = C(X_0) - C(X(t)) - \beta \log \left[\frac{\beta - C(X(t))}{\beta - C(X_0)} \right],$$

where $C(X_0) = c(0) = \beta - \lambda_0$. Taking the derivative with respect to X we find

$$C'(X) \frac{C(X)}{\beta - C(X)} = \rho. \tag{5.18}$$

It follows that $C'(0) = \infty$ because $C(0) = c(T) = 0$. Thus, the argument based on (5.14) to establish an asymptotic approach is not valid in this case.

Proof of Proposition 4(ii). We consider the case in which $\hat{X}$ is c-constrained, with $L(\hat{X}) \neq 0$ and $C(X) = M(\hat{X})$ for all X in some vicinity of $\hat{X}$ and $M(\hat{X}) \in \partial\mathcal{C}$ is determined by the c-constraint, while $|g_X(\hat{X}, M(\hat{X}))| < \infty$. We assume, without loss of generality, that $\hat{X} = \bar{X}$ so that the process increases towards $\hat{X}$ with $g(\cdot, \cdot) > 0$. Thus, $g_X(\hat{X}, M(\hat{X}))$ is negative and finite in this case. Consider the vicinity $\hat{X} - X < \varepsilon$ for some $\varepsilon > 0$ and write

$$g(X, C(X)) = g(X, M(\hat{X})) = g(\hat{X}, M(\hat{X})) + g_X(\tilde{X}, M(\hat{X}))[X - \hat{X}]$$

for some $\tilde{X} \in (X, \hat{X})$, hence

$$0 < g(X, C(X)) = [g_X(\hat{X}, M(\hat{X})) + O(\varepsilon)][X - \hat{X}] < -2g_X(\hat{X}, M(\hat{X}))[\hat{X} - X].$$

Repeating the arguments used to derive (5.14), we obtain

$$T - t_\varepsilon > \frac{1}{-2g_X(\hat{X}, M(\hat{X}))} \int_{X_\varepsilon}^{\hat{X}} \frac{dx}{\hat{X} - x},$$

and the integral diverges for every $X_\varepsilon < \hat{X}$, hence $T = \infty$. The case $\hat{X} = \underline{X}$ with $g(\cdot, \cdot) < 0$ and $g_X(\cdot, \cdot) > 0$ is treated in the same manner. $\square$

Consider, for example, the water resource management problem of the previous section with the specification $R(X) = \bar{X}^2 - X^2$ and $Z(\bar{X}) > u'(0)$. As shown above, the high unit cost of water extraction leaves no room for profitable exploitation and the stock approaches the upper bound $\bar{X}$ with $c = 0$ hence the steady state is c-constrained. Under this policy, the state evolution is governed by $\dot{X} = \bar{X}^2 - X^2$ which is readily integrated, yielding

$$X(t) = \bar{X}\frac{(\bar{X} + X_0)\exp(2\bar{X}t) - (\bar{X} - X_0)}{(\bar{X} + X_0)\exp(2\bar{X}t) + (\bar{X} - X_0)},$$

hence $T = \infty$ in agreement with the proposition. Observe that $C'(\bar{X}) = 0$ even though the resource is not essential at this c-constrained steady state, in contrast to the divergence of $C'(\cdot)$ for non-essential resources at their s-constrained boundaries. This difference underlies the different characterization of the corresponding arrival times.

6 Concluding comments

Two questions come up in any intertemporal planning problem: the first concerns the location of the final destination; the second deals with how long it takes to get there. The straightforward way to address these questions is to solve the underlying optimization problem and examine the long-term behavior of the state processes under the optimal policy. This approach may be quite cumbersome as it requires the parametric specification of all functions involved and often (when closed-form solutions are not available) resorts to numerical techniques. The alternative approach developed here addresses these questions for single-state processes via a simple algebraic method which avoids the solution of the underlying dynamic optimization problem.

The proposed method introduces the function $L(\cdot)$ of the state variable, which is used to formulate necessary conditions for optimal steady states. In many cases of interest, the necessary conditions narrow down the list of candidate steady states to a singleton, and the optimal state process converges to this unique state from any initial point. If several candidates meet the conditions, the optimal policy may depend on the initial state. The final choice, then, is determined by comparing the objective corresponding to each candidate.

The value obtained by the $L(\cdot)$-function at an optimal steady state determines whether the corresponding time of approach is finite or infinite. Steady states at which $L(\cdot)$ vanishes are always approached asymptotically. Otherwise, the time of approach depends on whether the constraints on the action (control) are binding at this state and on whether the resource is essential (i.e., the marginal benefit and $L(\cdot)$ diverge at the steady state) or non-essential (hence $L(\cdot)$ is finite).

Applying the method to a generic class of resource management models, we show how it can locate the candidates for an optimal steady state and, to each, characterize the time of approach under the optimal policy. The procedure can be similarly applied to any intertemporal planning problem that can be cast as a single-state, infinite-horizon, autonomous model.

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