AN APPLICABLE LINEAR PROGRAMMING MODEL OF INTER-TEMPORAL EQUILIBRIUM

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The problem of consumption over time in a world of certainty in which funds can be borrowed and lent, has been treated in a general fashion by Hirshleifer [6], whose analysis is an extension of the Fisherian theory. Basically, Hirshleifer's model consists of the consumer's time preference function in its most general form and a return-on-investment function. Of these only the second, of course, lends itself to direct application.

Baumol and Quandt (BQ) [1] presented a linear programming reformulation of Hirshleifer's model. While very constructive, their approach is not practical since it assumes knowledge of the utility function (and that it is linear). Further sources of illumination are provided by the studies of Charnes, Cooper and Miller (CCM) [3] and Ophir [10] who, ignoring consumption and using a profit function as their objective in place of the utility function, demonstrated the richness of information obtainable from a linear programming approach.

Two types of attempts to introduce consumption into multiperiod linear programming, without impairng applicability i.e., without resorting to unmeasurable utility functions, should be mentioned. The first, used by Loftsgard and Heady [8], was to impose predetermined consumption outlays which are thus independent of income, interest rates, etc. Basically, this is the CCM model with a neutral « tax » on the program. The second approach reintroduces consumption via the Keynesian, linear, consumption function. While feasible from the standpoint of applicability, it raises a set of problems which Boehlje and White [2] and Cocks and Carter [4] failed to realize. The essence of these problems lies with the fact that consumption, which constitutes no part of the objective, plays the role of an income tax, and this leads to a « distorted » solution. The distorsion reveals

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itself in two major ways: first, the internal rate of return will be undervalued; second, the pricing of production factors will be incorrect. As long as the programmed unit is a family firm, this may not pose a grave problem, since the program can be physically implemented regardless of the pricing system. When dealing with a sector or a region of a free economy, however, central imposition of the program is not usually possible, and the guidence is provided by the price system. An incorrect one will hardly serve the purpose.

The objective of this article is to present an applicable scheme for the proper use of the Keynesian consumption function, avoiding incorrect pricing. The core of the argument is a proposition proved in Section B, which yields the correct programming procedure. In section A, we briefly review some results obtained in the absence of consumption, illuminating in the process some points which seem to be of particular interest. Section C reviews an application of the model to the planning of the agricultural sector in a region in Israel.

A. *Inter-Temporal Equilibrium*

A detailed mathematical formulation of the model would involve extremely cumbersome notation. It thus seems easier to describe the model in stages. Consider a period of T years for which a program of production and investment is to be constructed. The economic unit involved can both borrow to finance its operations, and lend if it has no better alternatives, and is interested in maximizing its net worth (equity assets), ω, at the programming horizon. The interest rate charged on money borrowed during year t, i_t, is assumed to exceed the rate paid on deposists, r_t. Since it is assumed that the unit may borow unlimited sums at the going rate, one may view all loans as one-year loans. This approach does not cause any loss of generality and simplifies the exposition.

Problem I is to find non-negative x^t, z^t, $t = 1, 2, \dots, T$ (T being the programming horizon) which maximize ω subject to

$$(1) \quad \begin{bmatrix} k^{11} & m^{11} & 0 & 0 \cdots\cdots & & & 0 \\ A^{11} & B^{11} & \Phi & \Phi \cdots\cdots & & & 0 \\ k^{21} & m^{21} & k^{22} & m^{22} \cdots\cdots & & & 0 \\ \Phi & B^{21} & A^{22} & B^{22} \cdots\cdots & & & 0 \\ & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & & & & \\ 0 & m^{T1} & 0 & m^{T2} \,.. & k^{TT} & m^{TT} & 0 \\ \Phi & B^{T1} & \Phi & B^{T2} \,.. & A^{TT} & B^{TT} & 0 \\ 0 & s^1 & 0 & s^2 \,.. & p^T & s^T & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ z^1 \\ x^2 \\ z^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z^T \\ z^T \\ \omega \end{bmatrix} \leqq \begin{bmatrix} \mu_1 \\ q^1 \\ \mu_2 \\ q^2 \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \mu^T \\ q^T \\ \Omega \end{bmatrix}$$

In (1), matrices are denoted by capital letters, vectors by lower case letters and scalars by Greek letters; a null matrix and null vector of any dimension are denoted by Φ and 0, respectively, and the real zero is 0; transposition indicators are omitted, since only the last column of the matrix in (1), the variable vector and the right hand side, contain column vectors.

The matrix in (1), except for the last column, is composed of production, financial and investment activities. The set of columns having k^{tt} as its first non-zero row, describes production and financial operations, which can be undertaken in year t. Here, k^{tt} is the vector of cash requirements, A^{tt} is the matrix of input coefficients of production factors and k^{t+1t} is the revenue vector. The only exception are borrowing activities, which have the element (-1) in k^{tt} and $(1 + i_t)$ in k^{t+1t}.

An investment activity is defined as an operation which augments availability of production resources at some time after its initiation. The set of columns having m^{tt} as its first non-zero row, represents investment projects which can commence in year t. The matrix $B^{\tau t}$, $\tau = t, \ldots, T$ will have positive elements for inputs required and negative ones for outputs forthcoming.

We denote by p^T the vector whose elements represent contributions to (if negative) or claims against (if positive) terminal wealth, which result from production or financial operations in year T. Similarly, s^t is the vector of value residues contributed to terminal wealth by investment projects, which started in year t.

The vectors x^t and z^t are, respectively, the levels of activation of production and investment activities. The scalar μ_1 is the initial endowment of cash and q^1 plays the same role for real production factors. For $t > 1$, μ_t and q^t represent *exogenously* supplied resources. For some t, we may have $\mu_t = 0$, while q^t will be, in general, the non-obsolete remainder of q^1. Finally, Ω is the value of the non-obsolete portion of q^1 at the horizon.

In order to indicate the main properties of the solution to Problem I, we associate with μ_t, Ω and q^t the shadow prices λ_t, λ_{T+1} and u^t, respectively. The solution to Problem I will be denoted by $\{(\bar{x}^t, \bar{z}^t, \bar{\omega}, \bar{\lambda}_t, \bar{u}^t), t = 1, \ldots, T, \bar{\lambda}_{T+1}\}$. Also, for convenient reference, we refer to the set of columns in (1) having k^{tt} and m^{tt} as their first non-zero rows, as A^t and B^t, respectively.

1. The shadow price $\bar{\lambda}_t$ is the value to the program of an additional cash unit made available in year t, in terms of terminal wealth. That is, $\bar{\lambda}_t$ is the value of an additional dollar in year t, compounded

to the horizon. Thus, a dollar added at the horizon is worth one dollar. This can be seen, assuming a non-trivial solution, from the fact that at optimum $\overline{\lambda}_{T+1} = 1$.

The annual discount factor ρ_t, which is the program's annual equilibrium rate of return, is computed by

$$(2) \qquad \overline{\rho}_t = \frac{\overline{\lambda}_t}{\overline{\lambda}_{t+1}} - 1 \qquad t = 1, \ldots, T$$

so that

$$(3) \qquad \overline{\lambda}_t = \prod_{\tau = t}^{T} (1 + \overline{\rho}_\tau)$$

Inspection of the dual equations corresponding to B^t shows, that if in year t money is borrowed, $\overline{\rho}_t = i_t$, while if lending takes place, $\overline{\rho}_t = r_t$. In general,

$$(4) \qquad r_t \leq \overline{\rho}_t \leq i_t ,$$

since money can always be lent in the absence of superior activities, or borrowed in unlimited amounts (¹).

The same rate of return applies to all activities. For instance, let $a^t{}_j$ be any column of A^t such that $\overline{x}^t{}_j > 0$. Then by duality,

$$k_j{}^{tt}\, \overline{\lambda}_t + a_j{}^{tt}\, \overline{u}^t + k_j{}^{t+1t}\, \overline{\lambda}_{t+1} = 0 .$$

It follows, that

$$(5) \qquad \frac{\overline{\lambda}_t}{\overline{\lambda}_{t+1}} = \frac{- k_j{}^{t+1t} - (1/\overline{\lambda}_{t+1})\,(a_j{}^{tt}\, \overline{u}^t)}{k_j{}^{tt}}$$

The right hand side of (5) is a « natural » definition of the rate of return, while the left hand is, by (2), $1 + \overline{\rho}_t$. Note the division by $\overline{\lambda}_{t+1}$ in the numerator of (5). It is a result of the fact that $\overline{u}^t$ is a vector of compounded shadow prices, and λ_{t+1} rediscounts them to year t.

The rates of interest discussed above are money rates. It is, of course, most convenient to take money as the numeraire. From a strict theoretical point of view, however, this is an arbitrary convention. Rates of interest can be calculated in terms of any good. The alternative rates will not, in general, be equal to the money rate (²).

(¹) See Hirshleifer [6].
(²) See Malinvaud [9 p. 146].

The interpretation we gave to our model is dynamic, intertemporal. However, one could conceive of an instantaneous programming problem that will be formally identical to ours. It is only the meaning we attributed to the magnitudes and indices that made the model into an inter-temporal program. Bearing this point in mind helps to overcome some of the difficulties in interpretation.

3. It is very rare that long range programs are followed to the last year. In most cases changing economic circumstances will force re-planning. Multiperiod programming is undertaken in order to optimize short-rum actions, taking into account their long-run effect. It should therefore be interesting to investigate the effects of changing the length of the time span to which the program refers. To this end a comparison between a single-period and a multiperiod analysis is carried out. It will be shown that if the parameters in the single-period program are appropriately specified. the single-period program will yield a solution identical to the first peiiod of the multiperiod solution.

Consider the problem of finding non-negative x^1, z^1, which maximize $\omega\lambda_1$ subject to

$$(6) \qquad \begin{bmatrix} k^{11} & m^{11} & 0 \\ A^{11} & B^{11} & 0 \\ p & s & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ z^1 \\ \omega_1 \end{bmatrix} \leqq \begin{bmatrix} \mu_1 \\ q^1 \\ \Omega_1 \end{bmatrix}$$

Here, ω_1 is equity wealth at the end of the first year; Ω_1 is the end-of-year 1 value of the non-obsolete part of q^1; p and s are equivalent to p^T and s^T in (1). Letting $(x^{1*}, z^{1*}, \omega^*_1)$ be the optimal solution to the first year subproblem of Problem I, one can prove easily that if

$$(7) \qquad \lambda = \bar{\lambda}_2$$

$$(8) \qquad s = \Sigma^T_{t=2} \; (m^{t1} \; \bar{\lambda}_t + \breve{B}^{t1} \; \bar{u}^t) + s^1$$

where $\backsim$ denotes transposition, then

$$(x^{1*}, z^{1*}) = (\bar{x}^1, \bar{z}^1)$$

The last finding amounts to a restatement of the recursive nature of dynamic production processes: (3) given the appropriate prices, the economy can move from one period to the next, optimizing short run behavior within each period and, at the same time, following the long-rum optimal path. This is also a demonstration of Pontryagin's Maximum Principle (see e. g. Dorfman [4]).

(3) In discussing this point we benefited considerably from Jorgenson's « Lecture Notes on Capital Theory », Hebrew University, 1967 (mimeo).

The above discussion has practical implications centered on the terminal value of assets. These values are, as is evident from (8), the streams of income generated by these assets beyond the programming horizon. It also follows from (8) that in order to evaluate these streams correctly, one has to use the elements of $\bar{u}^t$ which, by (5), contain a compunding factor. For years beyond the horizon the correct compounding factor is not known and can only be guessed. One is thus liable to introduce mistakes which are compounded by each other. Hence, it stands to reasson that the closer the programming horizon — the larger the error, and hence its effects on the first year solution are more profound.

B. *Consumption*

Consumption in year t , c_t , is introduced via the Keynesian consumption function,

$$c_t = \alpha + \beta y_t , \qquad t = 1 , \ldots , T - 1 \tag{9}$$

where y_t is income in year t , to be defined below, and α , $0 < \beta < 1$, are parameters. Consumption in year T is not explicitly introduced — it is contained in the terminal wealth.

In order to formulate the new programming problem, the elemnts comprising net income, y_t , have to be defined. Two principal categories of income — cash income from production and financial activities, and income in the form of appreciation of assets — are distinguished. Income vectors in the first category, g^t , are defined by

$$g^t = - k^{t+1t} - k^{tt} \tag{10}$$

Note, that for a lending activity j , $g^t{}_j = r_t$, while if j is a borrowing activity, $g^t{}_j = - i_t$. The rationale underlying the definition in (10) is established as follows: assume for the sake of exposition that $\bar{x}^t > 0$. Then by duality,

$$\mathrm{k}^{tt}\, \bar{\lambda}_t + \tilde{A}^{tt}\, \bar{u}^t + \mathrm{k}^{t+1t}\, \bar{\lambda}_{t+1} = \underline{0} \tag{11}$$

Using the definition in (10), it follows from (3) and (11) that

$$g^t = [\tilde{A}^{tt}\, \bar{u}^t / \prod_{\tau = t+1}^{T} (1 + \bar{\rho}_\tau)] + \bar{\rho}_t\, k^{tt} \tag{12}$$

The right-hand side of (12) is the current value return to equity assets employed in A^t , and thus constitutes a natural definition of net income.

Income due to appreciation (if positive) or depreciation (if negative) of assets, is the difference in the value of these assets between successive years. Let $w^{\tau t}$, $\tau = t, \ldots T$, be the value-vector of assets whose construction began in year t. Then $w^{\tau t}$ is defined

$$(13) \qquad w^{\tau t} = -(1/\bar{\lambda}_{\tau+1})\,[\sum_{\theta=\tau+1}^{T} (m^{\theta t}\,\bar{\lambda}_\theta + \breve{B}^{\theta t}\,\bar{u}^\theta) + s^t]$$

In view of the role played by $\bar{\lambda}_t$ and $\bar{u}^t$, (13) is the net stream of benefits produced by investment projects from τ onwards, discounted to τ.

Appreciation (or depreciation) is now defined by

$$(14) \qquad v^{\tau t} = w^{\tau t} - w^{\tau-1 t} - m^{\tau t}, \qquad \tau = 1, \ldots, T \qquad w^{ot} = \underline{0}$$

In order to establish the logic of the definition in (14), assume $\bar{z}^t > 0$. Then

$$(15) \qquad \sum_{\theta=t}^{T} (m^{\theta t}\,\bar{\lambda}_\theta + \breve{B}^{\theta t}\,\bar{u}^\theta) + s^t = \underline{0}$$

Dividing (15) by $\bar{\lambda}_t$ and $\bar{\lambda}_{\tau+1}$ and using the resulting equations together with (13) and (14), one obtains

$$(16) \quad v^{\tau t} = (1/\bar{\lambda}_{\tau+1})\,\breve{B}^{\tau t}\,\bar{u}^\tau + \bar{\rho}_\tau\,m^{\tau t} + (\bar{\rho}_\tau/\bar{\lambda}_\tau)\sum_{\theta=t}^{\tau-1} (m^{\theta t}\bar{\lambda}_\theta + \breve{B}^{\theta t}\bar{u}^\theta)$$

The first term on the right hand side of (16) represents the change in value due to truncation of the income stream by one period; the second term is the return on cash investments during year τ and the third term expresses returns to equity assets invested in the project up to, and excluding, year τ.

As equation (16) indicates, appreciation and depreciation are functions of the optimal solution and thus cannot be known when the problem is set up. This fact prohibits us from direct application of the proposition proved below which, nevertheless, is of economic interest and quite useful, as will be indicated.

Rewriting (9) in the constraint form

$$(17) \qquad -c_t + \beta y_t \leq -\alpha,$$

Problem II is to find non-negative x^t, z^t, c^t which maximize

$$(18) \qquad f = \sum_{t=1}^{T-1} \delta_t\, c_t + \omega$$

subject to

$$
(19)\qquad
\begin{bmatrix}
k^{11} & m^{11} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
A^{11} & B^{11} & 0 & \Phi & \Phi & \cdots & \cdots & \cdots & \cdots & 0 \\
\beta g^{1} & \beta v^{11} & -1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
k^{21} & m^{21} & 1 & k^{22} & m^{22} & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \beta v^{T-1,1} & 0 & 0 & \beta v^{T-1,2} & \cdots & -1 & 0 & 0 & 0 \\
0 & m^{T1} & 0 & 0 & m^{T2} & \cdots & 1 & k^{TT} & m^{TT} & 0 \\
\Phi & B^{T1} & 0 & \Phi & B^{T2} & \cdots & 0 & A^{TT} & B^{TT} & 0 \\
0 & s^{1} & 0 & 0 & s^{2} & \cdots & 0 & p^{T} & s^{T} & 1
\end{bmatrix}
\begin{bmatrix}
x^{1} \\ z^{1} \\ c_{1} \\ x^{2} \\ \cdot \\ \cdot \\ \cdot \\ c_{T-1} \\ x^{T} \\ z^{T} \\ \omega
\end{bmatrix}
\leqq
\begin{bmatrix}
\mu_{1} \\ q^{1} \\ -\alpha \\ \mu^{2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_{T} \\ q^{T} \\ \Omega
\end{bmatrix}
$$

where δ_t are as yet unspecified parameters.

Let Problem II*a* be Problem II with

$$(20)\qquad \delta_t = 0 \qquad\qquad t = 1, \ldots, T-1$$

and denote the optimal solution, and the shadow prices associated with it, by $\{x^t, z^t, c^t, \omega\}$ and $\{\lambda_t, u^t, \eta_t\}$, respectively, where $\{\eta_t\}$ are the shadow prices of the consumption constraints. Assuming

$$(21)\qquad c_t > 0, \qquad\qquad t = 1, \ldots, T-1$$

it follows that

$$(22)\qquad \eta_t > 0,$$

since consumption plays the role of a mere income tax. This imposes a burden whose magnitude, per marginal dollar, is given by the values of η_t. The distortion which results from the imposition of consumption is best reflected in the fact, that the internal rate of return to which the system now adjusts itself is *net* of consumption expenditures. This can be best seen if one assumes, for simplicity, that in year t borrowing takes place. Then, by duality, we have instead of (2)

$$\frac{\lambda_t}{\lambda_{t+1}} = 1 + (1-\beta)\, i_t \equiv 1 + \rho_t$$

so that $\rho_t = (1-\beta)\,\bar{\rho}_t < \bar{\rho}_t$. As a result, one has instead of (5)

$$(5a)\qquad \frac{\lambda_t}{\lambda_{t+1}} = \frac{-k_j^{t+1t} - (1/\lambda_{t+1})\,(a_j^{tt}\, u^t)}{k_j^{tt}} - \beta\,\frac{g_j^{t}}{k_j^{tt}}$$

where the last term in (5*a*) represents the effect of consumption on the internal rate of return. It indicates, that the solution « tries to avoid » activities with a high ratio of income (or revenue) to expenditure.

The elimination of these difficulties can be achieved only through a « correct » selection of the δ_t values. Technically, the selection procedure can be effected via a search scheme involving parametric programming. The search procedure is terminated when a set $\{\delta_t^*\}$ and a corresponding primal-dual solution $\{x^{t*}, z^{t*}, c^{t*}, \omega^*, \lambda_t^*, u^{t*}, \eta_t^*\}$ are reached such that

$$(23) \qquad \delta^*_t = \lambda^*_{t+1}, \qquad t = 1, \ldots, T-1$$

$$(24) \qquad -c^*_t + \beta y^*_t = -\alpha, \qquad t = 1, \ldots, T-1$$

Looking at the consumption column in (19), it is obvious that

$$(25) \qquad -\eta^*_t + \lambda^*_{t+1} \geq \delta_t, \qquad t = 1, \ldots, T-1$$

Thus, an immediate consequence of (23) is

$$(26) \qquad \eta^*_t = 0, \qquad t = 1, \ldots, T-1$$

Equations (24) and (26) imply that we are looking for a so-called degenerate solution.

Under (23) one can show, that if $\{\bar{\lambda}_t, \bar{u}^t\}$ in (16) is replaced with $\{\lambda^*_t, u^{t*}\}$, then (3) and (5) hold for λ^*_t and by (26) consumption is no longer a burden to the system.

Before continuing with the analysis, some remarks are in order. As is well known, the optimum quantities consumed by an individual who maximizes a Fisherian utility function are such that the marginal rates of substitution in consumption between successive years equal the respective marginal rates of return to assets. Our solution is thus consistent with the individual's equilibrium. This consistency should, however, be put in the appropriate perspective: the Keynesian consumption function *cannot* be derived from a general Fisherian utility function. That is, there is no theoretical basis for suggesting that individual consumption behavior can be described by the Keynesian function. On the other hand, that function does describerather well aggregate behavior and the weights attached to consumptlon outlays in our solution, which are derived from aggregate behavior, also satisfy the optimality condition for the individual. These weights, the δ_t values, are equal, by (23), to the marginal contributions of funds reinvested in the program (1).

It is shown in [7], that the search procedure referred to above is finite. Practically speaking, however, it may take considerable time

(1) Note, that we have arrived at a BQ-type model, except that now the coefficients in the « utility function » are based on the observable consumption function.

and be quite expensive. Thus, the objective of the analysis from here on is to suggest instead a simpler procedure. It is based on the relation between the solution of Problem II*a* and the solution which satisfies (26), a relation which is derived below, avoiding the tedious search procedure.

Suppose, then, that Problem II*a* has been solved under the stipulation that in (16), $\{\bar{\lambda}_t\ \bar{u}_t\}$ is replaced with $\{\lambda_t^*\ u^{t*}\}$; i.e., it is assumed for the time being that the sought solution values are known in advance, so that appreciation and depreciation can be calculated without errors. Let D be the basic matrix of the solution. Noting that the problem is cast in the so-called revised form, assume for convenience that the first row of D^{-1} is the « pricing vector » (the shadow price vector). Further, let $\breve{h} = (1\ \lambda_1\ \breve{u}^1\ \lambda_2\ \breve{u}^2 \ldots \lambda_T\ \breve{u}^T)$, $\breve{e} = (\eta_1\ \eta_2 \ldots \eta_{T-1})$. Then D^{-1} may be written as

$$D^{-1} = \begin{bmatrix} \breve{h} & \breve{e} \\ D^{11} & D^{12} \\ D^{21} & D^{22} \end{bmatrix}$$

where D^{22} contains the elements common to the consumption rows and columns.

Next, let $\breve{d} = (\delta_1\ \delta_2 \ldots \delta_{T-1})$ and compute d^o from

$$\breve{d}^o D^{22} = -\breve{e} \tag{27}$$

Insert $d = d^o$ in the objective (18), without resolving the problem. This will give rise to a new set of shadow prices, $(h^o\ e^o)$. For $d = \underline{0}$, we have

$$(\breve{h}\ \breve{e}) = (1\ \breve{\underline{0}}\ \breve{\underline{0}})\ D^{-1}, \tag{28}$$

and for $d = d^o$,

$$(\breve{h}^o\ \breve{e}^o) = (1\ \breve{\underline{0}}\ \breve{d}^o)\ D^{-1} \tag{29}$$

From (28) and (29),

$$(\breve{h}^o\ \breve{e}^o) = (\breve{h}\ \breve{e}) + (0\ \breve{\underline{0}}\ \breve{d}^o)\ D^{-1} \tag{30}$$

which implies, together with (27),

$$e^o = \underline{0}, \tag{31}$$

which is equivalent to (26). That (24) is satisfied by c_t is obvious. Moreover, by (21) and (31)

$$\delta^o{}_t = \lambda^o{}_{t+1} \tag{32}$$

which is equivalent to (23). It is thus evident, that if the solution of Problem IIa is optimal with respect to d^o, then (30) indeed provides a simple way for calculating the correct shadow prices. The optimality of the solution is established below.

Equation (30) gives us the usual sensitivity analysis technique to solve for h^o. It can be verified, that in this case the relation between h and h^o is given by

$$(33) \qquad \lambda_t = \lambda^o{}_t \prod_{\tau = t}^{T-1} \left[1 - \frac{\beta \rho_\tau{}^o}{1 + \rho_t{}^o}\right] \qquad t = 1, \ldots, T-1$$

$$(34) \qquad u^t = (1 - \beta)\ u^{to} \prod_{\tau = t+1}^{T-1} \left[1 - \frac{\beta \rho_t{}^o}{1 + \rho_\tau{}^o}\right] \qquad t = 1, \ldots, T-2$$

$$(35) \qquad \lambda_T = \lambda^o{}_T$$

$$u^T = u^{To}$$

$$(36) \qquad u^{T-1} = (1 - \beta)\ u^{(T-1)o}$$

where $\rho^o{}_t$ is computed from (2) with $\lambda_t = \lambda^o{}_t$.

Proposition: If $v^{\tau t}$ in (16) is computed using $\{\lambda^{*}{}_t, u^{t*}\}$, then D is an optimal basis for Problem II with $d = d^*$.

Proof: To prove the proposition, one must show that the pricing vector $(h^o\ e^o)$ and its dot product with any column of the matrix in (19), are non-negative. Starting with the former, it follows from (33) that

$$(37) \qquad \lambda_t = \lambda^o{}_t \left(1 - \frac{\beta \rho^o{}_t}{1 + \rho^o{}_t}\right) \frac{\lambda_{t+1}}{\lambda^o{}_{t+1}}$$

From (2) and (37),

$$\lambda_t = [(1 - \beta)\ \lambda^o{}_t + \beta \lambda^o{}_{t+1}]\ \frac{\lambda_{t+1}}{\lambda^o{}_{t+1}}$$

which gives, rarranging terms,

$$(38) \qquad (1 - \beta)\ \frac{\lambda^o{}_t}{\lambda^o{}_{t+1}} = -\beta + \frac{\lambda_t}{\lambda_{t+1}}$$

Taking now the lending activity of year t and noting that in view of (21) $\eta_t = \lambda_{t+1}$, we have

$$(39) \qquad \frac{\lambda_t}{\lambda_{t+1}} \geq 1 + (1 - \beta)\ r_t > 1$$

Letting ρ_t satisfy

(40) $$\frac{\lambda_t}{\lambda_{t+1}} = 1 + (1 - \beta)\,\rho_t\,,$$

it follows from (38), (39) and (40) that

(41) $$\frac{\lambda^o{}_t}{\lambda^o{}_{t+1}} = 1 + \rho_t \equiv 1 + \rho^o{}_t > 1$$

This implies, together with $\beta < 1$,

(42) $$1 - \frac{\beta \rho^o{}_t}{1 + \rho^o{}_t} > 0$$

Using (35), (42) and applying (33) recursively, it follows that

(43) $$\lambda^o{}_t > 0\,, \qquad\qquad t = 1\,, \ldots\,, T$$

Using now (36), (42) and applying (34) recursively we also have

(44) $$u^{to} > 0\,, \qquad\qquad t = 1\,, \ldots\,, T$$

which, recalling (31), concludes the first part of the proof.

For any submatrix A^t, we have by the optimality of the solution to Problem II*a*

(45) $$\xi\,(x^t) \equiv k^{tt}\,\lambda_t + \bar{A}^{tt}\,u^t - \beta\,(k^{t+1t} + k^{tt})\,\lambda_{t+1} + k^{t+1t}\,\lambda_{t+1} \geqq 0$$

Using (31), (33) and (34) together with (45), we find

(46) $$k^{tt}\,\lambda^o{}_t + \bar{A}^{tt}\,u^{to} + k^{t+1t}\,\lambda^o{}_{t+1} = (1 - \beta)\;\xi\,(x^t) \geqq 0\,.$$

By the same reasoning underlying (45), we have for any submatrix B^t

(47) $$\xi\,(z^t) \equiv \sum_{\tau = t}^{T} (m^{\tau t}\,\lambda_\tau + \bar{B}^{\tau t}\,u^\tau) + \beta \sum_{\tau = t}^{T-1} v^{\tau t}\,\lambda_{\tau+1} + s^t \geqq 0$$

Using now (3), (16), (33), (34), (35), (36) and (46), it is not difficult to verify that

(48) $$\xi\,(z^t) = \sum_{\tau = t}^{T} (m^{\tau t}\,\lambda^o{}_\tau + \bar{B}^{\tau t}\,u^{\tau o}) + s^t\,,$$

and since $\xi\,(z^t)$ is non-negative, so is the right-hand-side of (48). Q.E.D.

The economic rationale underlying the property just established is, that the combination of activities which contributes most to terminal wealth, contributes most to consumption and terminal wealth provided these contributions are correctly measured.

Correct computations of appreciation and depreciation, necessary for the validity and direct applicability of the above proposition, cannot be effected at the formulation stage of the problem, since they involve solution values. In application, net income elements are computed by common accounting procedures, and will usually differ from the correct values. This will result in a situation in which investment activities with lower appreciation (higher depreciation) values will contribute more to the terminal wealth than to consumption as compared with investment actvities with higher appreciation or lower depreciation values. Thus, the primal solution to Problem II*a* will not, in general, remain unaffected by the insertion of d^o in (18). It is, however, our experience that applying the proposition, despite the practical shortcomings, reduces very considerably the computations involved in achieving a solution satisfying (23) and (24). This procedure was applied in the practical example illustrated in the next Section.

C. *Illustration*

The model discussed in Section *B* was applied to the agricultural sector of a region in nothern Israel. The program spans ten years and comprises a total of 360 production and investment activities, 9 consumption, 55 financial and 60 other activities - 484 in all. The constraint set consists of 459 constraints, of which 190 relate to limited production factors.

Some of the results are given in Tables 1 and 2. Table 1 contains part of the solution to Problem II*a*. From this solution d^o was computed in the manner indicated by (27) and Problem II was resolved with $d = d^o$. Although the second solution did not satisfy (31), the elements of e^o were so small that no further iterations seemed required. As could be expected, the two primal solutions are somewhat different, owing to errors in appreciation and depreciation of assets. In particular, most consumption elements in the second solution are higher than the corresponding ones in the first solution.

A few comments regarding the numerical results are in order. First, the high rates of interest, up to 13%, are not too high in a country where annual inflation rates are between 5% and 10%. Next, the impression that consumption is being programmed should not stay unamended. What is programmed is the amount of funds diverted from the production to the consumption sector. Some of these funds may be invested in durable consumer goods and thus need not increase

TABLE 1. - *Solution of Problem IIa.*

Year t	λ_t	η_t	c_t IL. 10^6	Sugar beet acres	Planting pears acres	One year credit IL. '000
1	1.258	1.252	16.057	650	495	—
2	1.252	1.236	17.234	650	173	48
3	1.236	1.221	16.772	650	142	—
4	1.221	1.205	15.974	650	108	433
5	1.205	1.190	15.100	650	23	2674
6	1.190	1.175	16.717	553	—	5107
7	1.175	1.160	17.232	354	—	5806
8	1.160	1.144	17.712	260	—	6335
9	1.144	1.130	18.657	170	—	4558
10	1.130	—	—	—	—	969

TABLE 2. - *Second solution of Problem II.*

Year t	δ^0_t	λ^0_t	η^0_t	ρ^0_t	c^0_t IL. 10^6	Sugar beet acres	Planting pears acres	One year credit IL. '000
1	2.977	3.150	.024	.058	16.525	650	650	—
2	2.634	3.001	.024	.13	17.146	650	650	1051
3	2.352	2.658	0.0	.12	16.386	650	625	1271
4	2.081	2.352	.001	.13	15.354	650	100	3767
5	1.842	2.082	0.0	.13	16.776	650	66	7957
6	1.630	1.842	0.0	.13	17.872	650	—	15262
7	1.443	1.630	0.0	.13	19.422	650	—	21123
8	1.277	1.443	0.0	.13	20.264	650	—	17894
9	1.129	1.277	.001	.13	20.871	649	—	13986
10	—	1.130	—	.13	—	—	—	9496

monotonically. This clearly merits a separate treatment, but falls without the scope of the present discussion.

Related to the last remark is the relatively low rate of annual consumption increase which proceeds at an average pace of 3% in the second solution. It is particularly low in Israel, where a rate of 8% is not uncommon. This is probably due to the fact that we did not take into account technological progress.

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